Theorem 1. Even-order convergents form an increasing and odd-order convergents a decreasing sequence. Also every odd-order convergent is greater than any even-order convergent. For any \( k, l \in \mathbb{N} \)
\[
\frac{p_0}{q_0} < \frac{p_2}{q_2} < \ldots < \frac{p_{2k}}{q_{2k}} < \frac{p_{2k+1}}{q_{2k+1}} < \ldots < \frac{p_3}{q_3} < \frac{p_1}{q_1}.
\]

Proof. Recall that \( a_j \in \mathbb{N} \) and so \( a_j \geq 1 \). Given that \( q_0 = 1, q_1 = a_1 \) one can use the recurrence relation
\[
q_k = a_k q_{k-1} + q_{k-2}
\]
with an induction argument to see that \( q_k > 0 \) for all \( k \geq 0 \). Now consider the second of the Casoratians
\[
\frac{p_{k-2}}{q_{k-2}} - \frac{p_k}{q_k} = (-1)^{k-1} \frac{a_k}{q_k q_{k-2}}.
\]
When \( k \) is odd we see
\[
\frac{p_k}{q_k} < \frac{p_{k-2}}{q_{k-2}}
\]
so the odd convergents form a decreasing sequence. Conversely when \( k \) is even the opposite case occurs
\[
\frac{p_k}{q_k} > \frac{p_{k-2}}{q_{k-2}}
\]
and the even convergents form a increasing sequence. To see that any odd convergent is greater than every even one we use the first Casoratian
\[
\frac{p_{k-1}}{q_{k-1}} - \frac{p_k}{q_k} = (-1)^k \frac{1}{q_k q_{k-1}}.
\]
Consider the \( 2k \)-th convergent and the \( 2l+1 \)-th convergent taking \( l > k \) without any loss of generality. Now from the above we have
\[
\frac{p_{2l+1}}{q_{2l+1}} > \frac{p_{2l}}{q_{2l}}
\]
and from the ordering of the even convergents
\[
\frac{p_{2l}}{q_{2l}} > \frac{p_{2l-2}}{q_{2l-2}} > \ldots > \frac{p_{2k}}{q_{2k}}.
\]
\( \Box \)

Theorem 2. For arbitrary \( k (1 \leq k \leq n) \)
\[
[a_0; a_1, a_2, \ldots, a_n] = \frac{p_{k-1} r_k + p_{k-2}}{q_{k-1} r_k + q_{k-2}}.
\]
Proof. We see that
\[
[a_0; a_1, a_2, \ldots, a_k, a_{k+1}, \ldots] = [a_0; a_1, a_2, \ldots, a_{k-1}, r_k]
\]
\[
= \frac{r_k p_{k-1} + p_{k-2}}{r_k q_{k-1} + q_{k-2}}.
\]
□

Lemma 1. The mediant of two fractions \(\frac{a}{b}\) and \(\frac{c}{d}\)
\[
\frac{a+c}{b+d}
\]
always lies between them in value.

Proof. For definiteness suppose \(\frac{a}{b} \leq \frac{c}{d}\). Then
\[
bc - ad \geq 0,
\]
and consequently
\[
\frac{a+c}{b+d} - \frac{a}{b} = \frac{bc - ad}{b(b + d)} \geq 0,
\]
\[
\frac{a+c}{b+d} - \frac{c}{d} = \frac{ad - bc}{(b+d)d} \leq 0.
\]
□

Theorem 3. Let \(x\) be an arbitrary positive real number having a regular continued fraction representation with partial quotients \(a_k\) and numerator and denominator convergents \(p_k, q_k\) respectively. For arbitrary \(k \geq 0\)
\[
\frac{1}{q_k(q_{k+1} + q_k)} < \left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}.
\]

Proof. Taking the upper bound first we know from Theorem 2 that
\[
x = \frac{r_{k+1}p_k + p_{k-1}}{r_{k+1}q_k + q_{k-1}}.
\]
so that the absolute error is
\[
\left| x - \frac{p_k}{q_k} \right| = \frac{r_{k+1}p_k + p_{k-1} - p_k}{r_{k+1}q_k + q_{k-1}}.
\]
\[
= \frac{r_{k+1}(p_k q_k - p_k q_k) + q_k p_{k-1} - p_k q_{k-1}}{q_k(r_{k+1}q_k + q_{k-1})}.
\]
\[
= \frac{1}{q_k(r_{k+1}q_k + q_{k-1})}.
\]
Now \(a_k = \lfloor r_k \rfloor\) so that \(r_k \geq a_k\). Consequently
\[
r_{k+1}q_k + q_{k-1} \geq a_{k+1}q_k + q_{k-1} = q_{k+1}.
\]
and
\[
\left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}.
\]
For the proof of the lower bound consider the mediant of \( p_k/q_k \) and \( p_{k+1}/q_{k+1} \). As \( p_{k+1}/q_{k+1} \) lies closer to \( x \) than \( p_k/q_k \) then the mediant of the two lies closer to \( p_k/q_k \) than \( x \):

\[
\left| \frac{x - p_k}{q_k} \right| > \left| \frac{p_k + p_{k+1}}{q_k + q_{k+1}} - \frac{p_k}{q_k} \right| = \left| \frac{p_{k+1}q_k - p_kq_{k+1}}{q_k(q_k + q_{k+1})} \right| = \frac{1}{q_k(q_k + q_{k+1})}.
\]

\[\square\]

**Theorem 4.** For \( k \geq 2 \)

\[ q_k \geq 2^{(k-1)/2}. \]

**Proof.** From the recurrence relation we see

\[ q_k = a_k q_{k-1} + q_{k-2} \geq q_{k-1} + q_{k-2} \geq q_{k-1}, \]

because \( q_{k-2} > 0 \) and therefore \( q_k \geq q_{k-1} \). But we can do better than this.

\[
q_k \geq q_{k-1} + q_{k-2} \\
\geq 2q_{k-2} \\
\geq 2^2 q_{k-4} \\
\vdots \\
\geq \begin{cases}
2^{(k-1)/2} q_1 & k \text{ odd} \\
2^{k/2} q_0 & k \text{ even} \\
\end{cases} \\
\geq 2^{(k-1)/2}.
\]

\[\square\]

**Exercise 1.** The continued fraction for \( \frac{\sqrt{5}-1}{2} \) has for its \( n \)-th convergent the ratio

\[
\frac{p_n}{q_n} = \frac{F_n}{F_{n+1}},
\]

where \( F_n \) is the \( n \)-th Fibonacci number. Prove

\[
\left| \frac{x - p_n}{q_n} \right| \to n \to \infty \frac{1}{\sqrt{5}q_n^2}
\]
Theorem 5 (Lagrange). For all irrational numbers \( x \) there exists an infinite number of rational approximations \( p/q \) such that

\[
\frac{|x - p|}{q} < \frac{1}{\sqrt{5q^2}}.
\]