1. Consider a less than full rank model with two factors. Factor 1 has two levels, and factor 2 has 3 levels. We take 2 samples from each possible combination of factor levels. We denote the response variable from the $k$th sample from the combination of factors with the first factor at level $i$ and the second factor at level $j$ to be $y_{ijk}$. We also denote the overall mean by $\mu$, and assume that each level of each factor adjusts this overall mean by a certain amount: $\tau_i$ for the $i$th level of factor 1, and $\beta_j$ for the $j$th level of factor 2.

(a) Express $y_{ijk}$ according to $\mu$, $\tau_i$, $\beta_j$, and an error term.
(b) Write down the linear model in matrix form.

2. Let

$$A = \begin{bmatrix} 1 & 2 & 5 & 2 \\ 3 & 7 & 12 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix}. $$

(a) Show that $r(A) = 2$.
(b) Find a conditional inverse for $A$.

3. Show that $A = A(A^T A)^{-1} A^T A$. You may use the result that if $A^T A = 0$, then $A = 0$. (Hint: Consider the matrix $A - A(A^T A)^{-1} A^T A$.)

4. It is known that toxic material was dumped into a river that flows into a large salt-water commercial fishing area. We are interested in the amount of toxic material (in parts per million) found in oysters harvested at three different locations in this area. A study is conducted and the following data obtained:

<table>
<thead>
<tr>
<th>Site 1</th>
<th>Site 2</th>
<th>Site 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>19</td>
<td>22</td>
</tr>
<tr>
<td>26</td>
<td>15</td>
<td>26</td>
</tr>
</tbody>
</table>

(a) Write down the linear model in matrix form.
(b) Write down the normal equations.
(c) Find a conditional inverse for $X^T X$.
(d) Find a solution for the normal equations.

5. In a manufacturing plant, filters are used to remove pollutants. We are interested in comparing the lifespan of 5 different types of filters. Six filters of each type are tested, and the time to failure in hours is given in the dataset (on the website) filters (in csv format).

(a) Use the read.csv function to read the data. Then convert the type component into a factor.
(b) Construct X and y matrices for this linear model.
(c) Using the algorithm given in the lecture slides, find a conditional inverse for $X^T X$.
(d) Use ginv to find another conditional inverse for $X^T X$.
(e) Find two solutions for the normal equations.
(f) Express one of your solutions in terms of the other.