1. Show that the gamma distribution is an exponential family.

**Solution:** The gamma distribution with shape $\nu > 0$ and rate $\lambda > 0$ has log density

$$\log f(x; \nu, \lambda) = (\nu - 1) \log(x) - \lambda x + \nu \log(\lambda) - \log(\Gamma(\nu))$$

$$= x(-\lambda/\nu) \log(\lambda) \frac{1}{\nu} - \nu \log(1/\nu) + (\nu - 1) \log(x) - \log(\Gamma(\nu))$$

Put $\theta = -\lambda/\nu$ and $\phi = 1/\nu$ then we have

$$\log f(x; \nu, \lambda) = \frac{x\theta - \log(-1/\theta)}{\phi} - \phi \log(\phi) + \left(\frac{1}{\phi} - 1\right) \log(x) - \log(\Gamma(1/\phi))$$

This is in the form of an exponential family, with

$$b(\theta) = \log(-1/\theta)$$
$$a(\phi) = \phi$$
$$c(x, \phi) = -\log(\phi) + (1 - \phi) \log(x) - \phi \log(\Gamma(1/\phi))$$

Note that with this parameterisation we have $\theta < 0$ and $\phi > 0$.

For the canonical link $g$ we have $g(\mu) = \theta$. Here $\mu = \nu/\lambda = -1/\theta$, so $g(x) = -1/x$. (Note that in practice people tend to use the inverse link $x \mapsto 1/x$ rather than $x \mapsto -1/x$, because it is convenient to keep things positive.) The variance is $\nu/\lambda^2 = \phi \mu^2 = a(\phi) v(\mu)$. That is, the variance function is $v(\mu) = \mu^2$.

2. Prove that if a random variable $X$ has density

$$f(x; \theta, \phi) = \exp \left[ \frac{x\theta - b(\theta)}{a(\phi)} + c(x, \phi) \right]$$

Then

$$E X = b'(\theta)$$
$$\text{Var} X = b''(\theta) a(\phi).$$

**Hint:** show that for any likelihood $L$ we have

$$E \frac{\partial \log L}{\partial \theta} = 0$$
$$E \frac{\partial^2 \log L}{\partial \theta^2} = -E \left( \frac{\partial \log L}{\partial \theta} \right)^2.$$
**Solution:** first note that for any likelihood $L$ we have

\[
\mathbb{E} \frac{\partial \log L}{\partial \theta} = \int \frac{\partial \log L(\theta; x)}{\partial \theta} L(\theta; x) dx 
\]

\[
= \int \frac{1}{L(\theta; x)} \frac{\partial L(\theta; x)}{\partial \theta} L(\theta; x) dx 
\]

\[
= \int \frac{\partial L(\theta; x)}{\partial \theta} dx 
\]

\[
= \frac{\partial}{\partial \theta} \int L(\theta; x) dx 
\]

\[
= \frac{\partial}{\partial \theta} 1 = 0. 
\]

Similarly

\[
\mathbb{E} \frac{\partial^2 \log L}{\partial \theta^2} = \mathbb{E} \frac{\partial}{\partial \theta} \left( \frac{1}{L(\theta; x)} L'(\theta; x) \right) 
\]

\[
= \mathbb{E} \frac{L'(\theta; x)^2 - L''(\theta; x)}{L(\theta; x)^2} + \int \frac{L''(\theta; x)}{L(\theta; x)} L(\theta; x) dx 
\]

\[
= \mathbb{E} \frac{\partial \log L}{\partial \theta}^2 + \frac{\partial^2}{\partial \theta^2} \int L(\theta; x) dx 
\]

\[
= \mathbb{E} \frac{\partial \log L}{\partial \theta}^2 . 
\]

Applying the first result to $f$ we have

\[
0 = \mathbb{E} \frac{\partial}{\partial \theta} \left[ \frac{X \theta - b(\theta)}{a(\phi)} + c(X, \phi) \right] 
\]

\[
= \mathbb{E} \frac{X - b'(\theta)}{a(\phi)} 
\]

whence $\mathbb{E}X = b'(\theta)$.

Applying the second result we have

\[
-\mathbb{E} \left( \frac{X - b'(\theta)}{a(\phi)} \right)^2 = \mathbb{E} \frac{\partial^2 \log L}{\partial \theta^2} 
\]

\[
- \frac{\text{Var } X}{a(\phi)^2} = \mathbb{E} \frac{\partial}{\partial \theta} \frac{X - b'(\theta)}{a(\phi)} 
\]

\[
= \mathbb{E} \frac{-b''(\theta)}{a(\phi)} = -\frac{b''(\theta)}{a(\phi)} 
\]

whence $\text{Var } X = b''(\theta)a(\phi)$, as required.