1. Prove the following versions of Fatou’s lemma and the Dominated Convergence Theorem. Here everything is $\mathcal{F}$-measurable, and $\mathcal{G}$ is a sub-$\sigma$-algebra of $\mathcal{F}$.

**cFATOU** If $X_n \geq 0$ then
\[ \mathbb{E}[\lim \inf X_n | \mathcal{G}] \leq \lim \inf \mathbb{E}[X_n | \mathcal{G}] . \]

**cDOM** If $|X_n| \leq V$ for all $n$, $\mathbb{E}V < \infty$, and $X_n \to X$ a.s., then
\[ \mathbb{E}[X_n | \mathcal{G}] \to \mathbb{E}[X | \mathcal{G}] \text{ a.s.} \]

2. (Williams E10.1 Pólya’s urn)
   At time 0 an urn contains 1 black ball and 1 white ball. At each time 1, 2, … a ball is chosen at random from the urn and is replaced together with a new ball of the same colour. Thus, just after time $n$ there are $n+2$ balls in the urn, of which $B_n+1$ are black, where $B_n$ is the number of black balls chosen up to and including time $n$.

(a) Let $M_n = (B_n+1)/(n+2)$ be the proportion of black balls in the urn just after time $n$. Prove that (relative to a natural filtration which you should specify) $M = \{M_n\}$ is a martingale.

(b) Prove that $\mathbb{P}(B_n = k) = 1/(1+n)$ for $0 \leq k \leq n$. What is the distribution of $\Theta := \lim M_n$?

(c) Prove that for $0 < \theta < 1$, $N^\theta = \{N^\theta_n\}$ is a martingale, where
\[ N^\theta_n = \frac{(n+1)!}{B_n!(n-B_n)!} \theta^{B_n} (1-\theta)^{n-B_n} . \]

3. (Williams E10.8 Bayes’ urn)
   A random number $\Theta$ is chosen uniformly between 0 and 1, and a coin with probability $\Theta$ of heads is minted. Toss the coin repeatedly, and let $B_n$ be the number of heads in $n$ tosses. Prove that $\{B_n\}$ has the same probabilistic structure as the sequence $\{B_n\}$ in the previous question.

4. Let $f : [0,1] \to \mathbb{R}$ be Lipschitz, that is, suppose that for some $K < \infty$ and all $x, y \in [0,1]$, $|f(x) - f(y)| \leq K|x - y|$.
   Denote by $f_n$ the simplest piecewise linear function agreeing with $f$ on $\{k2^{-n} : k = 0, 1, \ldots, 2^n\}$.
   Set $M_n = f'_n$. Show that $M_n$ converges a.s. and in $\mathcal{L}^1$ and deduce that $f$ is the indefinite integral of a bounded function.