Lemma 3 Let \( \mathcal{P} \) be the partition above, let \( X^* \) and \( N^* \) be the interval length and family size of a ‘uniformly’ chosen point, and let \( J^* \) be the position of the chosen interval within its family. If \( \mu = \sum_x x p(x) \) and \( m = \mathbb{E} X_{i,j} \) are finite then with probability 1, for \( 1 \leq l \leq n \)

\[
P(N^* = n, J^* = l, x \leq X^* < x + dx \mid \mathcal{P}) = \frac{np(n)}{\mu m} \frac{x}{n} dF(x).
\]

Note that \( X^* \) will have atoms at the same points as \( X_{i,j} \).

\textbf{Proof} Let \( T_k = \sum_{i=1}^k \sum_{j=1}^{N(i)} X_{i,j} \) and let \( \mathcal{P}_k \) be the partition of \([0,T_k]\) given by \( X_{1,1}, \ldots, X_{k,N(k)} \). Given \( \mathcal{P}_k \), choose \( x \) uniformly on \([0,T_k]\) and let \( X^*_k \) and \( N^*_k \) be the interval length and family size of \( x \). Let \( S_k(n) = \#\{i : 1 \leq i \leq k, N(i) = n\} \) then sending \( k \to \infty \)

\[
P(N^*_k = n, J^* = l, X^*_k \leq x \mid \mathcal{P})
\]

\[
= \sum_{i=1}^k \sum_{j=1}^n I_{\{n\}}(N(i)) I_{\{l\}}(j) I_{[0,x]}(X_{i,j}) \frac{X_{i,j}}{T_k}
\]

\[
= \sum_{i=1}^k S_k(n) \frac{1}{k} \sum_{i=1}^k \frac{1}{S_k(n)} I_{\{n\}}(N(i)) X_{i,i} I_{[0,x]}(X_{i,i})
\]

\[
= \frac{1}{\mu m} p(n) \mathbb{E} X_{i,j} I_{[0,x]}(X_{i,j}) \text{ with probability 1}
\]

\[
= \frac{np(n)}{\mu} \frac{1}{n} \int_0^x y dF(y).
\]

Differentiating w.r.t. \( x \) gives the result.

By integrating/summing out the other terms, one can easily show that the marginal distributions of \( N^* \) and \( X^* \) are given by

\[
P(N^* = n) = \frac{np(n)}{\mu};
\]

\[
P(x \leq X^* < x + dx) = \frac{x}{m} dF(x).
\]

Similarly, the conditional distribution of \( J^* \) given \( N^* \) is given by

\[
P(J^* = l \mid N^* = n) = \frac{1}{n} \text{ for } 1 \leq l \leq n.
\]