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Combinatorial Curvature of Cellular Complexes

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Abstract

One of the primary issues with the use of differential geometry is the difficulty in calculating metrics, and the properties which follow, such as curvature. Robin Forman has published a great deal of work which attempts to provide a combinatorial analogue to curvature. This project investigateds Forman's work, along with a proof of a major result, namely Myer's Theorem.

Despite his progress, Forman's work is ultimately flawed in its limited applicability, as few examples satisfying his strong conditions can be constructed. Additionally, some major theorems in smooth differential geometry, such as the Gauss-Bonnet theorem, cannot in fact coexist with his definition. In response to this, an alternate formulation of combinatorial curvature is developed, which addresses these issues.

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Chapter 1:

Introduction

Differential geometry is a rich area that has heavily influenced later disciplines, such as topology. When applied to the study of abstract topological manifolds, differential geometry was able to re-introduce some of the structure that was lost through this abstraction. In addition to providing a metric and inner product, a differential geometric approach enabled considerations with regard to more significant properties of manifolds, such as curvature and vector fields. In the Riemannian setting, both of these properties have been found to be indispensable in limiting the complexity of the topology which may be observed. Morse theory (see, for example, [17]) provided a method of finding efficient cell structures for generic manifolds, through analysis of critical points of real-valued functions on these manifolds. Vector fields lead to bounds on the orders of the homology groups of a manifold, through the Morse Inequalities which consider critical (rest) points of smooth vector fields. Finally, curvature has become integral in the classification conjecture of Thurston, a geometric interpretation which incorporates, and provides context for, the Poincaré Conjecture. Prior to this, Myers proved a theorem [19] which utilised a special class of vector fields called Jacobi fields to show that the fundamental group of a positively curved space is finite. All of these theorems were proved using Riemannian (smooth) methods.

Despite being an area with a rich history, differential geometry has been a subject in which it is difficult to make progress. This is largely due to the difficulty posed by calculation of metrics on general manifolds, which is necessary for an investigation of properties such as curvature. In addition, choosing an appropriate metric on the cells of a decomposition of a manifold can lead to difficulties. To this end, Robin Forman has written several papers (see, for example, [4], [6], [7]) which attempt to construct a discrete analogue of the major theorems in differential geometry, with varying degrees of success. Apart from his work, this area has been largely ignored in general, despite the fact that the discrete approach has two distinct advantages: first, the cell decomposition itself becomes the metric, with or without weights on the cells; and second, the properties are easily calculable by simple computer programs, even for decompositions with extremely large numbers of cells.

This paper begins with the basic definitions and properties of Forman's discrete analogues, largely as they appeared in [4] and [7]. We then give some examples, which are lacking in Forman's work, along with the proof of some important and (mostly) non-trivial results. The third chapter is primarily concerned with the proof of Myers' Theorem using Forman's curvature definitions, and a proof methodology which is parallel to that used by Myers in his original paper, [19]. Finally, we address some concerns with regards to Forman's methodology and consider an alternative formulation which may be used.

1.1 Cells and Complexes

We will consider an abstract manifold $\mathbb{M}$, which is composed of *cells*. We say that a cell $\sigma^{(p)} \in \mathbb{M}$ is called a p-cell if it is homeomorphic to an open p-ball. Cells are attached to each other via *boundary maps*, with the boundary of a p-cell being entirely composed of cells $\{\sigma^{(q)}\}_{q<p}$. Often, we will consider complexes of triangles or tetrahedra where the edges have been identified (triangulations) as they are simple to both understand and depict, though this is by no means necessary. Additionally, we are primarily concerned with closed (empty boundary), connected manifolds.

We begin with a formal definition of the main class of complexes we will use, namely CW-complexes.

Definition 1.1 (CW-complex) *[8] A CW-complex is a cell complex which is constructed in the following manner:*

(i) Begin with a discrete set X^0, whose points are considered 0-cells (or vertices).

(ii) Inductively form the n-skeleton X^n from X^{n-1} by attaching n-cells $\sigma_\alpha^{(n)}$ via maps $h_\alpha : S^{n-1} \to X^{n-1}$. This means that X^n is the quotient space of the disjoint union $X^{n-1} \coprod_\alpha \bar{\mathbb{B}}_\alpha^n$ under the identifications $x \sim h_\alpha(x)$ for $x \in \partial\bar{\mathbb{B}}_\alpha^n$.

(iii) Either stop at some finite stage, and set $X = X^n$, or continue indefinitely and set $X = \bigcup_n X^n$.

Additionally, we require the following two properties:

(i) Closure-finiteness: The closure of each cell meets only finitely many other cells; and

(ii) Weak topology: A set $A \subset X$ is open (closed) if and only if $A \cap X^n$ is open (closed) in X^n for all n.

Definition 1.2 *Let $\mathbb{M}$ be a CW-complex. Then we say that $\mathbb{M}$ is* regular *if for all $\alpha \in \mathbb{M}$,*

$$h_\alpha : \sigma^{(\dim\alpha)} \longrightarrow \mathbb{M}$$

maps $\sigma^{(\dim\alpha)}$ homeomorphically onto its image.

As most of our discussion of cells in complexes is concerned with 0-, 1- and 2-cells, for convenience we shall call them vertices, edges and faces (respectively) and typically denote them by v, e and f (also respectively).

Definition 1.3 *The* closure *of a cell* $\sigma^{(p)}$ *in a CW-complex* $\mathbb{M}$ *is the set of cells* $\bar{\sigma}^{(p)} = \{\alpha^{(q)} \in \mathbb{M} : \alpha \cap \varphi_\sigma(\sigma^{(p)}) \neq \emptyset$ *and* $q < p\}$. *Also, define the* boundary *of a cell as* $\partial\sigma^{(p)} = \bar{\sigma}^{(p)} \backslash \sigma^{(p)}$.

We also wish to define some notation. Given cells $\tau^{(q)}$, $\sigma^{(p)}$ with $q < p$, then we write $\tau \prec \sigma$ or equivalently $\sigma \succ \tau$ if and only if $\tau \in \partial\sigma$. Also, for any face f then sides(f) is $\#\{e \prec f\}$.

Sometimes it will be necessary to be more specific than the class of CW-complexes, and so we can consider instead *topological manifolds.* Here we will always assume that a manifold is also a CW-complex, with a given decomposition. We first define the *link* of a cell, and then the class of manifolds which will be considered here.

Definition 1.4 *Let* $\mathbb{M}$ *be an* n*-dimensional, regular CW-complex. For any* p*-cell* $\sigma^{(p)}$, *the* link *of* $\sigma^{(p)}$ *is the collection of cells formed by taking the boundaries of small balls transverse to the* p*-cell in each* n*-cell containing it.*

Definition 1.5 (Topological Manifold) *Let* $\mathbb{M}$ *be an* n*-dimensional, regular CW-complex. Then* $\mathbb{M}$ *is also a* manifold *if for every* p*-cell* $\sigma^{(p)}$ *the link of* $\sigma^{(p)}$ *is homeomorphic to* S^{n-p-1}. *Additionally,* $\mathbb{M}$ *is* closed *if it is compact and has empty boundary.*

To generalise our notion of cell-complexes, we introduce the concept of *weights.* A weight is a value attached to a cell, which could depend either on specific cells or some more general class. Here, we will only consider a special class of weights, namely a *standard set.*

Definition 1.6 *Let* $\mathbb{M}$ *be a CW-complex, where each cell* α *has been assigned a weight* w_α. *Then* $\{w_\alpha\}$ *is a* standard set of weights *if for every* α, *we have*

$$w_\alpha = \omega_1 . \omega_2^{(\dim \alpha)}$$

where ω_1, ω_2 *are positive constants.*

We now turn our attention to the arrangement of cells in a decomposition of some arbitrary manifold $\mathbb{M}$. The primary concern in this regard is the connections between edges. To facilitate meaningful discussion, we require some terminology:

Definition 1.7 *An edge* e' *is said to be a* 0-neighbour *of* e *if* $\bar{e} \cap \bar{e'} = v$ *for some* v *and* $\nexists f$ *such that* $e \prec f$ *and* $e' \prec f$. *Additionally, a* u-0-neighbour *of* e *is a 0-neighbour with* $\bar{e} \cap \bar{e}' = u$.

Definition 1.8 *Similarly, an edge e' is said to be a* 2-neighbour *of e if $\exists f$ such that $e \prec f$ and $e' \prec f$ and $\bar{e} \cap \bar{e}' = \emptyset$*

Thus an edge is a *0-neighbour* if it shares only a vertex, and a *2-neighbour* if they share a face. Collectively, we refer to 0- and 2-neighbours as *parallel* neighbours. In the case where two edges share both a vertex and a face, we say that the edges are *transverse*; and if they do not have any shared 0- or 2-cells, then we say that the edges are *distant*.

1.2 Curvature

We now turn our attention to the notion of curvature. Forman, in [7], defines the Ricci curvature of an edge as follows:

Definition 1.9 (Curvature) *For any edge e in a CW-complex of a manifold $\mathbb{M}$, the combinatorial Ricci curvature of e, $\mathrm{Ric}(e)$, is:*

$$\mathrm{Ric}(e) = \#(v \in \mathbb{M} : v \prec e) + \#(f \in \mathbb{M} : f \succ e) - \#(\textit{0-neighbours of } e) - \#(\textit{2-neighbours of } e)$$

Classically, curvature is defined at a point and in a direction, where Ricci curvature is defined as the average sectional curvature of planes through a point p and perpendicular to a direction $\mathbf{v}$, say. To explicitly define smooth Ricci curvature, we first need some elementary definitions:

Definition 1.10 *Let $\mathbb{M}$ be isometrically embedded in $\mathbb{R}^N$ for some N. Then the* scalar curvature *of a geodesic arc $\gamma(t)$, at a point $\gamma(0) = p \in \mathbb{M}$, is defined as*

$$\kappa_\gamma(p) = \frac{1}{\textit{radius of osculating circle}}$$

where the osculating circle is the circle which goes through p, is in the direction of $\dot{\gamma}$, and most closely approximates $\ddot{\gamma}$.

For any surface, the principal curvatures at p, often written κ_1, κ_2, are the maximum and minimum of the scalar curvature at that point. In fact, these are always orthogonal (see, for example [18] for a proof). Now:

Definition 1.11 (Gaussian Curvature) *Let Σ be a surface which is isometrically embedded in $\mathbb{R}^N$ for some N, with $p \in \Sigma$, and the principal curvatures at p being κ_1, κ_2. Then the* gaussian curvature *$\boldsymbol{\kappa}$ at p is $\boldsymbol{\kappa} = \kappa_1 \kappa_2$.*

Finally we can define the sectional curvature:

Definition 1.12 *For any plane* $\Pi(\mathbf{u}, \mathbf{v})$ *passing through some point* $p \in \mathbb{M}$*, the* sectional curvature *in the plane* Π *at* p *is the gaussian curvature of the surface* $\Sigma_\Pi = \{\textit{geodesics } \gamma : \ \gamma(0) = p, \ \dot{\gamma} \in \Pi\}$.

And thus we can define smooth Ricci curvature as follows:

Definition 1.13 *For any point* $p \in \mathbb{M}$ *and direction* $\mathbf{v}$*, then the (smooth)* Ricci curvature *at* p *in the direction of* $\mathbf{v}$ *is the average sectional curvature of all planes through* p *perpendicular to* $\mathbf{v}$*. It is sufficient to consider an orthogonal basis*

$$\{\mathbf{u_i} : \ \mathbf{u_i} \cdot \mathbf{u_j} = 0, i \neq j \ \textit{and} \ \mathbf{u_i} \cdot \mathbf{v} = 0 \ \forall \, i\}_{i \in [m]}$$

and then

$$\mathrm{Ric}(p, \, \mathbf{v}) = \frac{1}{\binom{m}{2}} \sum_{1 \leq i < j \leq m} \mathrm{Sec}(p, \, \Pi(\mathbf{u_i}, \mathbf{u_j})).$$

Thus it can be seen that sectional curvature provides more information than Ricci curvature, at the expense of simplicity. We note that positive (negative) sectional curvature for all planes through a point p guarantees that the Ricci curvature is positive (negative) at p, but the converse is false.

A parallel between the smooth and combinatorial definitions of Ricci curvature is not immediately apparent. In order to illuminate this, we have expanded upon Forman's definition to provide an analogue to sectional curvature which is consistent with the Ricci curvature in Definition 1.9:

Definition 1.14 *For any vertex* v *and edge* e *in a CW-complex of a manifold* $\mathbb{M}$*, the combinatorial sectional curvature at* v *in the direction of* e*,* $\mathrm{Sec}(v, e)$*, is:*

$$\mathrm{Sec}(v, e) = 2 + \#(f \in \mathbb{M} : f \succ e) - 2\#(\textit{v-0-neighbours of } e) - \#(\textit{2-neighbours of } e)$$

We now examine the correspondence between the two definitions, namely that :

Lemma 1.1 *For an embedded edge* e*, The combinatorial Ricci curvature of an edge is equal to the average of the sectional curvature of all vertices* $v \prec e$*, in the direction of* e*.*

Proof: Since e is embedded, then it has two distinct vertices, say u and v. We now seek to compute the average sectional curvature from each of these:

$$\begin{aligned}
\frac{1}{2}\left(\mathrm{Sec}(u,e)\right. &+ \left.\mathrm{Sec}(v,e)\right) \\
&= \frac{1}{2}\Big[2 + \#(f \in \mathbb{M} : f \succ e) - 2\#(u\text{-0-neighbours of } e) - \#(\text{2-neighbours of } e) \\
&\qquad +2 + \#(f \in \mathbb{M} : f \succ e) - 2\#(v\text{-0-neighbours of } e) - \#(\text{2-neighbours of } e)\Big] \\
&= \frac{1}{2}\Big[4 + 2\#(f \in \mathbb{M} : f \succ e) - 2\#(\text{2-neighbours of } e) - 2\#(u\text{-0-neighbours of } e) \\
&\qquad -2\#(v\text{-0-neighbours of } e)\Big] \\
&= 2 + \#(f \in \mathbb{M} : f \succ e) - \#(\text{2-neighbours of } e) - \#(\text{0-neighbours of } e) \\
&= \#(v \in \mathbb{M} : v \prec e) + \#(f \in \mathbb{M} : f \succ e) - \#(\text{2-neighbours of } e) \\
&\qquad -\#(\text{0-neighbours of } e) \\
&= \mathrm{Ric}(e).
\end{aligned}$$

This proves the lemma. □

Another important result which is neglected in Forman's work is the amount of information required to calculate the (combinatorial) curvature of a CW-complex. We have the simple result:

Lemma 1.2 *The combinatorial Ricci curvature of any edge in a CW-decomposition of a manifold* $\mathbb{M}$ *is only dependent upon the 2-skeleton of* $\mathbb{M}$.

Proof: Since we consider only edges and whether or not they share a face or a vertex, then the Ricci curvature is independent of all cells of dimension greater than 2. Thus it is only dependent upon the 2-skeleton of $\mathbb{M}$. □

This result, despite its triviality, enables a (slight) relaxation of the assumptions for our major result (Myers' Theorem, 3.1), namely that only the 2-skeleton of our manifold $\mathbb{M}$ need be quasiconvex (see Definition 1.22). Our final remark on curvature at this point is that the definition can be extended to a CW-complex in which all the cells $\alpha \in \mathbb{M}$ have been given weights w_α. This formula is extensively derived in [7], however we omit it here for brevity.

Definition 1.15 (Generalised Ricci curvature) *Let* $\mathbb{M}$ *be a CW-complex, where each cell* α *has been assigned a weight* w_α. *Then the Ricci curvature of an edge* $e \in \mathbb{M}$ *is*

$$\mathrm{Ric}(e) = w_e \left\{ \left[\sum_{f^{(2)} \succ e} \frac{w_e}{w_f} + \sum_{v^{(0)} \prec e} \frac{w_v}{w_e} \right] - \sum_{e' \neq e} \left| \sum_{\substack{f^{(2)} \succ e \\ f^{(2)} \succ e'}} \frac{\sqrt{w_e w_{e'}}}{w_f} - \sum_{\substack{v^{(0)} \prec e \\ v^{(0)} \prec e'}} \frac{w_v}{\sqrt{w_e w_{e'}}} \right| \right\}.$$

1.3 Functions on Cell Complexes

Now that the basic structures with which we are concerned have been established, we turn our attention to functions on these structures. In smooth differential geometry, vector fields are a useful tool for analysis, and as such a combinatorial analogue is desirable. Here we define such a function, along with its elementary properties. Following this, we define a class of function which generally behaves like a vector field, but which possesses important properties which make it invaluable in the proof of our major theorem (see Theorem 3.1); namely Jacobi fields. We begin with the combinatorial vector field:

Definition 1.16 *A* Combinatorial Vector Field V *on a CW-complex* $\mathbb{M}$ *is a map which pairs cells* $\{(\sigma, V(\sigma))\}$*, such that the following conditions hold:*

(i) Each cell $\sigma \in \mathbb{M}$ *appears in precisely one pair; and*

(ii) If $\sigma^{(p)}$ *is a p-cell, then either* $V(\sigma^{(p)})$ *is a* $(p+1)$*-cell, or* $V(\sigma^{(p)}) = \emptyset$*.*

A typical representation of a combinatorial vector field V (as can be found in [4], [6], [7]) is through drawing an arrow from a p-cell $\sigma^{(p)}$ to a $(p+1)$-cell $\tau^{(p+1)}$ if $\tau^{(p+1)} = V(\sigma^{(p)})$. If $\sigma^{(p)}$ is paired with $\emptyset$, then draw no arrow. A simple example on the 3-sphere is shown in Figure 1.1. In this depiction, each cell is unpaired, the head of an arrow, or the tail of an arrow. The first condition in Definition 1.16 ensures that a cell satisfies precisely one of these.

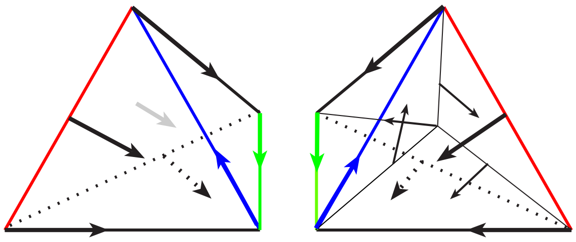

Figure 1.1: A combinatorial vector field on S^3.

In the smooth setting, a simple way to analyse vector fields is to examine *critical points* and *cycles*. Critical points are those points without a well-defined tangent vector in the vector field, while cycles are closed flow-lines under the vector field, with a flow line being that traced out over time where at each point the direction of the flow is defined by the vector field. Combinatorially, we can define appropriate analogues:

Definition 1.17 *[4] A* stationary point *or a* rest point *of a vector field V on a CW-complex $\mathbb{M}$ is any $\alpha \in \mathbb{M}$ such that $V(\alpha) = \emptyset$.*

Definition 1.18 *Let V be a combinatorial vector field on a CW-complex $\mathbb{M}$. Then a* combinatorial vector flow *of length k is a sequence of distinct cells $\{\alpha_1, V(\alpha_1), \alpha_2, \ldots, \alpha_k, V(\alpha_k)\}$ such that $V(\alpha_i) \succ \alpha_{i+1}$. Such a flow is also a k*-cycle *if $\alpha_1 \prec V(\alpha_k)$. We also write that a flow (or cycle) is of order p if each α_i is a p-cell.*

There are some important differences between combinatorial vector fields and their smooth analogues. First, there is no unique flow from a given point. In the smooth setting, this is true since at any point in time (along a flow), the tangent vector at that point is well-defined, and smoothly varying. Combinatorially, this cannot be true as Figure 1.2 demonstrates on a segment of a combinatorial vector field, with the flow (black arrows) continuing along either the blue or red arrows. However, since each α has a unique $V(\alpha)$, we can recover uniqueness of definition of some cycles (this is proved in Lemma 2.1).

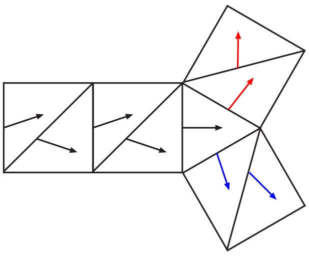

Figure 1.2: A combinatorial vector flow, with non-unique (degenerate) continuation.

Second, since a combinatorial vector field pairs cells with those of a dimension higher, there is no (obvious) way in which to form the 'negative' of a vector field. In the smooth case, this corresponds to reversing the vectors at each point, however this is not permitted combinatorially. To compensate for this, we can use the *dual decomposition*:

Definition 1.19 (Dual) *Let $\mathbb{M}$ be an n-dimensional CW-complex. Then the* dual *decomposition $\mathbb{M}'$ is defined as follows:*

1. *For each $(n-p)$-cell $\sigma_\alpha^{(n-p)} \in \mathbb{M}$, associate a p-cell $(\sigma_\alpha^{(p)})' \in \mathbb{M}'$.*
2. *For any two cells $(\sigma_\alpha^{(p)})'$ and $(\sigma_\beta^{(q)})'$ in $\mathbb{M}'$, with $p > q$, then $(\sigma_\beta^{(q)})' \prec (\sigma_\alpha^{(p)})'$ if and only if $\sigma_\beta^{(n-q)} \succ \sigma_\alpha^{(n-p)}$ where $\sigma_\beta^{(n-q)}$ and $\sigma_\alpha^{(n-p)}$ are the corresponding cells in $\mathbb{M}$.*

For simplicity, we will often write σ_α and σ'_α for the original and dual cells, respectively. The more explicit notation will only be used when we wish to stress the dimensions of the cells.

And then considering vector fields:

Definition 1.20 *Let V be a vector field on a CW-complex $\mathbb{M}$, with dual decomposition $\mathbb{M}'$. Then the* inverse vector field $-V$ *is defined as*

$$(-V)(\tau') = \sigma' \ \leftrightarrow \ \tau = V(\sigma).$$

The main problem with this approach is that this is defined *globally*, and hence there is no way to *locally* reverse a field, which is a property often used in Morse theory to 'cancel' pairs of critical points.

Combinatorial vector fields are investigated more thoroughly, along with the proof of some relevant lemmas, in Section 2.2.

Using a special class of vector fields known as *gradient vector fields*, Forman [4] was able to prove one of the key results of Morse theory (see, for example [17]), the Morse Inequalities. These provide a bound for the dimensions of the homology groups for an arbitrary manifold. A full proof can be found in [4], but will not be given here. Instead, we turn our attention to a different class of function, Jacobi fields.

Definition 1.21 *A* Combinatorial Jacobi Field *along a path $\gamma = v_0 e_1 e_2 \cdots e_k v_k$ is a map*

$$J : \{e_i\}_{i \in [k]} \longrightarrow \{\textit{2-cells}\}$$

such that

(i) $J(e_i) \succ e_i$ for all i; and

(ii) for every $i \in [k-1]$, $J(e_i)$ and $J(e_{i+1})$ share at least one edge other than e_i and e_{i+1} ($J(e_i) = J(e_{i+1})$ is permitted).

It is important to notice that while they have some similarities, combinatorial Jacobi fields are *not* combinatorial vector fields. Combinatorial Jacobi fields are investigated and examples are given in Section 2.4.

1.4 Properties of Complexes

In addition to consideration of both cell complexes and functional tools which operate upon them, it is necessary the properties of the complexes themselves. These can be included (if necessary) as conditions to theorems, and be used at key points of the proof. As will be seen later in our major result and its preliminaries (see Theorems 3.1-3.5) the key property we will require is *quasiconvexity*:

Definition 1.22 (Quasiconvexity) *Let* $\mathbb{M}$ *be a regular CW-complex. We say* $\mathbb{M}$ *is* quasiconvex *if for every two* $(p+1)$*-cells* $\sigma_1 \neq \sigma_2$*, with* $\bar{\sigma}_1 \cap \bar{\sigma}_2$ *containing a* p*-cell* τ*, then* $\bar{\sigma}_1 \cap \bar{\sigma}_2 = \bar{\tau}$*.*

The following result, which is asserted without proof by Forman in [7], allows us to consider all CW-complexes, after finding a quasiconvex cell-decomposition. Importantly, the proof is constructive, and hence gives an algorithm to find such a decomposition for any given CW-complex.

Lemma 1.3 *Every CW-complex admits a quasiconvex cell-decomposition.*

Proof: We shall construct a quasiconvex cell-decomposition from one which is not. It is sufficient to show that an arbitrary failure of the quasiconvexity condition can be rectified, as then the lemma can be reapplied until no failures remain, and the cell-decomposition is quasiconvex.

Take any p-cell, $\sigma_1^{(p)} \in \mathbb{M}$, which fails the quasiconvexity condition for at least one $\sigma_2^{(p)}$, that is $\tau \in (\bar{\sigma}_1^{(p)} \cap \bar{\sigma}_2^{(p)})$ but $(\bar{\sigma}_1^{(p)} \cap \bar{\sigma}_2^{(p)}) \neq \bar{\tau}$. Define the *stellate* subdivision of $\sigma_1^{(p)}$, $f^\star(\sigma_1^{(p)})$ as follows:

1. Embed a 0-cell, say v in $\sigma_1^{(p)}$;
2. For all $v_i^{(0)} \in \bar{\sigma}_1^{(p)}$, add an edge $e_i \in \sigma_1^{(p)}$, with endpoints v and $v_i^{(0)}$;
3. More generally, for all $\tau^{(q)} \in \bar{\sigma}_1^{(p)}$, add a $(q+1)$-cell, $\tau^{(q+1)}$, where $\partial\, \tau^{(q+1)}$ is the closure of $\{\text{added cells } \upsilon^{(q)} : \ \bar{\upsilon}^{(q)} \cap \bar{\tau}^{(q)} \neq \emptyset\}$;
4. Continue adding $(q+1)$-cells in this fashion, until $\sigma_1^{(p)}$ is divided into $\#\{\tau^{(p-1)} \prec \sigma_1^{(p)}\}$ distinct p-cells, with all cells in $\bar{\sigma}_1^{(p)}$ being identified as before; and
5. Leave all remaining cells unchanged.

We now seek to prove two claims:

Claim 1: *All introduced cells are quasiconvex with respect to each other.*

Assume by way of contradiction that there exists $\tau_1^{(q)} \neq \tau_2^{(q)}$ such that $\tau_1^{(q)}, \tau_2^{(q)} \in f^\star(\sigma_1^{(p)})$ and $\upsilon^{(q-1)} \in (\tau_1^{(q)} \cap \tau_2^{(q)})$ but $\bar{\upsilon}^{(q-1)} \neq (\bar{\tau}_1^{(q)} \cap \bar{\tau}_2^{(q)})$. Since each of $\tau_1^{(q)}$, $\tau_2^{(q)}$ were defined on separate $(q-1)$-cells in $\bar{\sigma}_1^{(p)}$, then they must share newly added cells. However, since each added cell was added with precisely those which intersect the original $(q-1)$-cells in $\bar{\sigma}_1^{(p)}$, then for $\upsilon^{(q-1)} \in (\tau_1^{(q)} \cap \tau_2^{(q)})$, we know that $\forall \varsigma \in \bar{\upsilon}^{q-1}$, $\varsigma \in (\tau_1^{(q)} \cap \tau_2^{(q)})$. Additionally, there can be no other cells (as at each stage the maximum number were introduced) and so $\bar{\upsilon}^{(q-1)} = \bar{\tau}_1^{(q)} \cap \bar{\tau}_2^{(q)}$, which is a contradiction and so the claim is true.

Claim 2: *$f^\star(\sigma_1^{(p)})$ is now quasiconvex with respect to the CW-complex, $f^\star(\mathbb{M})$.*

Since $\forall \tau_i^{(p-1)}, \tau_j^{(p-1)} \in \bar{\sigma}_1^{(p)}$ we have that $\tau_i^{(p-1)}$ and $\tau_j^{(p-1)}$ are in different p-cells of $f^\star(\sigma_1^{(p)})$, and furthermore $\bar{\tau}_i^{(p-1)}$ and $\bar{\tau}_j^{(p-1)}$ are the only original cells from $\mathbb{M}$ which are in these p-cells. Since Claim 1 holds, then $f^\star(\sigma_1^{(p)})$ is quasiconvex with respect to $f^\star(\mathbb{M})$, which proves the second claim, and hence the lemma. $\square$

Since the curvature is only dependent upon the 2-skeleton (see Lemma 1.2), then we can relax our assumptions slightly and only require that the 2-skeleton is quasiconvex. As we shall see in Lemma 3.4, this is vital for the proof of our main result.

Chapter 2:

Combinatorial Curvature: Results and Examples

The primary power of a combinatorial approach to a mathematical problem is its computability: combinatorial methods enable the development of algorithms which can calculate various properties of topological spaces. The usefulness of these algorithms is dependent upon the correspondence between the combinatorial and smooth approaches. This correspondence is not immediate, since a simplification occurs when shifting from the smooth to the discrete, so it is necessary to determine whether too much information has been discarded.

Here, we shall first give examples of combinatorial curvature calculations, along with an important failure of a classical theorem, namely that of Gauss-Bonnet. This is followed by an examination of vector fields with some examples, along with some results which are standard in continuous differential geometry.

2.1 Some Combinatorial Curvature Calculations

The primary purpose of this section is to provide some examples of combinatorial curvature examples, something which is largely neglected in Forman's work [7]. While he provides some simple examples in the plane, there is nothing more substantial or complex. In attempting to redress this, it readily becomes apparent that there is a relative paucity of examples of manifolds with positive curvature (in the smooth sense) which maintain this under Forman's definition (see Definition 1.9). Thus Forman's criterion is incredibly strong, which poses difficulties not only in generating large numbers of examples, but also in the application of any results to general manifolds. All the examples below, with the exception of $\mathbb{R}^n$, are known (classically) to have positive Ricci curvature in the smooth sense. Note that for brevity we refer to *parallel neighbours*, a category which includes both 0- and 2-neighbours.

2.1.1 The Real Plane, $\mathbb{R}^2$

For a first example, we examine the standard decomposition of the real plane into squares (see Figure 2.1), where the vertices are the integer lattice points, the edges are all of unit length between two vertices and the faces consist of connected regions of points with no integer coordinates. In this case, it is fairly simple to see (by the large degree of symmetry) that all edges are in an equivalent environment. Thus by examining the red edge (say e) in Figure 2.1, we can see that it has two neighbouring faces, two 0-neighbours (coloured green, see Definition 1.7) and two 2-neighbours (coloured blue, refer to Definition 1.8). Finally, the red edge has (as with all embedded edges) two distinct vertices. Thus:

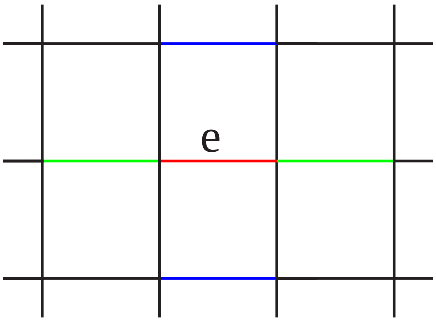

Figure 2.1: Rectangular decomposition of $\mathbb{R}^2$.

$$\begin{aligned}\mathrm{Ric}(e) &= \#\{v \prec e\} + \#\{f \succ e\} - \#\{\text{0-neighbours}\} - \#\{\text{2-neighbours}\} \\ &= 2+2-2-2 \\ &= 0\end{aligned}$$

and so the real plane is flat in a combinatorial sense, as with the smooth case.

A Generalisation to $\mathbb{R}^n$

We now seek to generalise the above result to (hyper)-cubic decompositions of $\mathbb{R}^n$. A (hyper)-cubic decomposition is analogous to that above, where the integer points are vertices, edges join vertices and are of length one, and faces consist of those points with exactly two non-integer coordinates. Recall that the curvature of a manifold $\mathbb{M}$ is only dependent upon the underlying 2-skeleton (see Lemma 1.2). Since we have a (hyper)-cubic decomposition, then we know the following:

(i) Each edge $e \in \mathbb{R}^n$ is embedded, and so $\#\{v \prec e\} = 2$;

(ii) Each face $f \in \mathbb{R}^n$ is square, which means that for every face there is a 2-neighbour and so $\#\{f \succ e\} = \#\{\text{2-neighbours}\}$; and

(iii) Since all edges are orthogonal, the only 0-neighbours are those which are parallel (in a geometric sense) to the edge being examined (namely e)

Thus we can conclude that:

$$\begin{aligned}\mathrm{Ric}(e) &= \#\{v \prec e\} + \#\{f \succ e\} - \#\{\text{0-neighbours}\} - \#\{\text{2-neighbours}\}\\ &= 2 - 2 + \#\{f \succ e\} - \#\{\text{2-neighbours}\}\\ &= 0\end{aligned}$$

and then $\mathbb{R}^n$ is flat under a (hyper)-cubic decomposition.

2.1.2 The 2-Sphere, S^2

We shall consider four different decompositions of the 2-sphere in order to highlight two important facts. First, we wish to examine the different kinds of parallel neighbours (namely 0- and 2-neighbours) which can occur; and second, to illustrate the fact that curvature (in Forman's sense) varies depending upon the decomposition of the manifold. This second fact is natural since in the smooth case the curvature depends upon the choice of metric. In the combinatorial setting, we can only count the number of cells — which then becomes the metric — and so the particular cell-decomposition chosen will determine the curvature. In particular, this means that subdivision (which is useful for ensuring that quasiconvexity is satisfied, see Lemma 1.3) *may* alter the curvature sufficiently so that it is no longer strictly positive.

Tetrahedral decomposition of S^2

We now consider the tetrahedral decomposition of the 2-sphere (refer to Figure 2.2): Since all edges are equivalent, without loss of generality, we consider the red edge e:

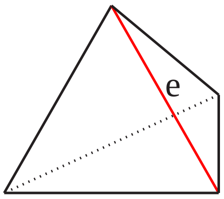

Figure 2.2: Tetrahedral decomposition of S^2.

$$\begin{aligned}\mathrm{Ric}(e) &= \#\{v \prec e\} + \#\{f \succ e\} - \#\{\text{parallel neighbours}\}\\ &= 2 + 2 - 0\\ &= 4.\end{aligned}$$

There are no 2-neighbours since each face is a triangle, and hence only has edges e' with $e' \prec f$ and $e' \succ v$ where $v \prec e$ for some v. Additionally, there are no 0-neighbours since each vertex has degree three, and there are two faces adjacent to an edge.

Cubic decomposition of S^2

If the 2-sphere is decomposed as a cube: Once again, all edges are equivalent, so consider the red edge, e:

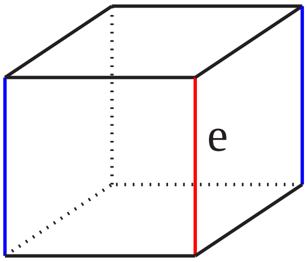

Figure 2.3: Cubic decomposition of S^2.

$$\begin{aligned}
\mathrm{Ric}(e) &= \#\{v \prec e\} + \#\{f \succ e\} - \#\{\text{parallel neighbours}\} \\
&= 2 + 2 - 2 \\
&= 2.
\end{aligned}$$

Both parallel neighbours in this case arise from the faces (that is, they are 2-neighbours of e), these edges are coloured blue in Figure 2.3.

Octahedral decomposition of S^2

The octahedral decomposition of the 2-sphere: Considering the red edge, e (as all edges are equivalent):

$$\begin{aligned}
\mathrm{Ric}(e) &= \#\{v \prec e\} + \#\{f \succ e\} - \#\{\text{parallel neighbours}\} \\
&= 2 + 2 - 2 \\
&= 2.
\end{aligned}$$

In this case, the parallel neighbours are 0-neighbours (coloured green in Figure 2.4). These arise since all vertices are of degree four.

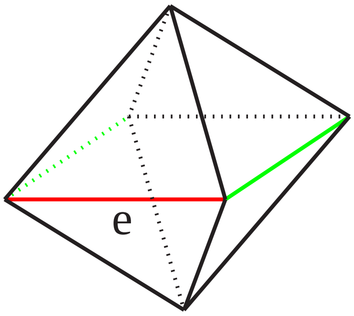

Figure 2.4: Octahedral decomposition of S^2.

Buckyball decomposition of S^2

The buckyball decomposition of S^2 gives our first indication of the problems posed by Forman's combinatorial curvature. Additionally, this highlights the strength of any condition requiring such a curvature to be strictly positive, as even though the particular choice of decomposition should change the curvature (as this defines our metric), in the smooth case this changes only the magnitude, but never the sign.

In the buckyball decomposition, we have two types of edges. For an edge between two hexagons (as with the blue edge in Figure 2.5):

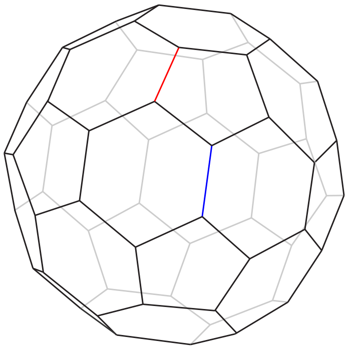

Figure 2.5: Buckyball decomposition of S^2.

$$\begin{aligned}
\mathrm{Ric}(e) &= \#\{v \prec e\} + \#\{f \succ e\} - \#\{\text{parallel neighbours}\} \\
&= 2 + 2 - 6 \\
&= -2.
\end{aligned}$$

And with edges between a pentagon and a hexagon, such as the red edge in Figure 2.5:

$$\begin{aligned}\mathrm{Ric}(e) &= \#\{v \prec e\} + \#\{f \succ e\} - \#\{\text{parallel neighbours}\}\\ &= 2+2-5\\ &= -1.\end{aligned}$$

It is clear that the curvature depends upon the particular CW-decomposition of the manifold which is chosen. Indeed, as can be seen from Section 2.1.2, even the sign of the curvature is not fixed.

2.1.3 The 3-Sphere, S^3

We will now consider S^3, with the cell decomposition as follows. Take two tetrahedra and identify the faces without any twisting. Now take one of the original tetrahedra, and subdivide it into four smaller tetrahedra, such that each tetrahedron contains precisely one face of the original tetrahedron (see Figure 2.6). This is a quasiconvex decomposition into five tetrahedra. We first consider an edge inside the subdivided tetrahedron, the red edge e in Figure 2.6, which has curvature

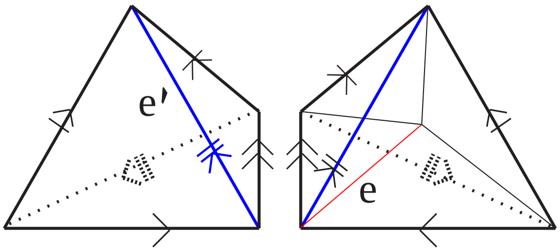

Figure 2.6: Five tetrahedron decomposition of S^3.

$$\begin{aligned}\mathrm{Ric}(e) &= \#\{v \prec e\} + \#\{f \succ e\} - \#\{\text{parallel neighbours}\}\\ &= 2+3-0\\ &= 5.\end{aligned}$$

If we now consider the blue edge e', then

$$\begin{aligned}\mathrm{Ric}(e') &= \#\{v \prec e'\} + \#\{f \succ e'\} - \#\{\text{parallel neighbours}\}\\ &= 2+3-0\\ &= 5.\end{aligned}$$

Thus for this decomposition of S^3, the combinatorial Ricci curvature is always five. If one examines this decomposition more closely, it is equivalent to considering S^3 as the boundary of a 4-simplex. This allows us to generalise the decomposition to S^n.

A Generalisation to S^n

If we consider S^n as the boundary of an $(n+1)$-simplex, then we can represent it as a hypergraph on $n+2$ vertices, say Γ. Then any proper subset $G \subsetneq \Gamma$ with $|G| = p$ corresponds to a $(p-1)$-cell in the simplex. Specifically, two vertices determine an edge (all of which are present), three vertices determine a face (all of which are also present), and so on. Then we know that for any edge, say e, we have $(n+2) - \#\{v \prec e\} = n$ faces adjacent to the edge; and since every such face is present, then there are no 0-neighbours to e. Finally, as this constitutes a triangulation of S^n, then there are no 2-neighbours, and hence no parallel neighbours. We calculate:

$$\begin{aligned}\mathrm{Ric}(e) &= \#\{v \prec e\} + \#\{f \succ e\} - \#\{\text{parallel neighbours}\}\\ &= 2 + n - 0\\ &= 2 + n \ > 0 \quad \text{for all } n.\end{aligned}$$

By construction, there exists a positively curved quasiconvex decomposition of S^n, as would be expected from smooth results.

2.1.4 Projective 3-Space, $\mathbb{RP}^3$

If we consider real projective 3-space, then it is reasonably simple to construct an example which has positive cuvature (see Figure 2.7), however this decomposition is not quasiconvex. Indeed, the best which could be constructed was one with non-negative (everywhere-zero, in fact) curvature.

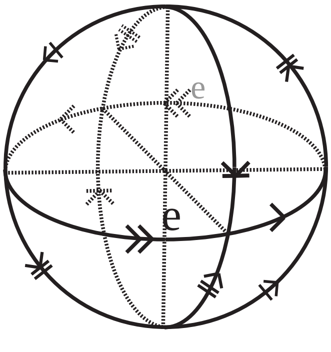

Figure 2.7: Eight Tetrahedron Decomposition of $\mathbb{RP}^3$ (positively curved, but not quasiconvex).

If we consider a “Rubik’s Cube” decomposition of $\mathbb{RP}^3$, as in Figure 2.8, where the opposite faces of the (large) cube have been identified with a 180° twist (as can be seen by the colour-coding in Figure 2.8), then we can see that all edges are equivalent, since each is surrounded by four distinct cubes. We can thus calculate the curvature

$$\begin{aligned}\mathrm{Ric}(e) &= \#\{v \prec e\} + \#\{f \succ e\} - \#\{\text{parallel neighbours}\} \\ &= 2 + 4 - (2 + 4) \\ &= 0,\end{aligned}$$

and so we have constructed a quasiconvex decomposition of $\mathbb{R}\mathcal{P}^3$ which is everywhere Ricci-flat.

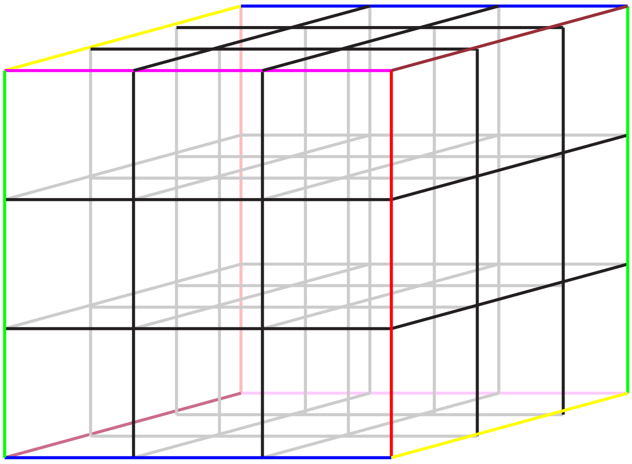

Figure 2.8: “Rubik’s Cube” decomposition of $\mathbb{R}\mathcal{P}^3$.

An Extension to $\mathbb{R}\mathcal{P}^n$

A key strength to the decomposition above is that for all n, we can use an analogous decomposition. Consider an n-dimensional hyper-cube, with opposite faces identified under rotation by 180°. Subdivide this into 3^n identical hyper-cubes, as with the “Rubik’s Cube” decomposition in Figure 2.8, then the curvature of any of the equivalent edges is

$$\begin{aligned}\mathrm{Ric}(e) &= \#\{v \prec e\} + \#\{f \succ e\} - \#\{\text{parallel neighbours}\} \\ &= 2 + 2^n - (2 + 2^n) \\ &= 0,\end{aligned}$$

and the combinatorial Ricci curvature is zero for all edges e.

2.1.5 Lens spaces

The simplest non-trivial lens spaces are $L(3,1)$, $L(4,1)$, $L_{5,1}$ and $L_{5,2}$. These last two are the first pair of non-homeomorphic spaces with the same fundamental groups, and hence provide an interesting example. All attempts to construct a quasiconvex decomposition, using both the method of Heegaard splittings and subdivision of triangulations, have failed to produce an example with positive (or even non-negative) curvature on all edges for any of these. In fact, we posit that it may be impossible to find such a decomposition, as discussed in Section 4.1.

2.2 Combinatorial Vector Fields

Here we seek to investigate combinatorial vector fields (see Definition 1.16) more thoroughly, along with an extensive treatment of an example to illustrate their basic properties. We will begin with an example. In Figure 2.9, a combinatorial vector field on S^3 is shown. For simplicity, the identifications have been omitted, however the full vector field, along with the identifications, can be found in Figure 1.1. There are two cycles, one of order zero (red arrows), and another of order one (green arrows). Additionally, the rest points of this vector field are precisely the blue 2-cells, and the 3-cells in the right-hand tetrahedron.

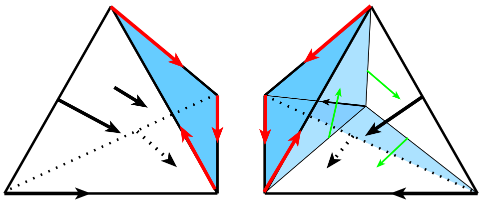

Figure 2.9: A combinatorial vector field on S^3 (identifications not shown).

Recall that combinatorial vector fields suffer from two principal difficulties: the lack of a local inverse and the non-uniqueness of flows. Examining Figure 2.10, it can be seen that two issues can cause non-uniqueness of a flow. The first can be rectified for flows of order $(n-1)$ in an n-dimensional manifold, as then each $(n-1)$-cell σ has at most two n-cell neighbours and hence the choice is unique (as one of them must be $V(\sigma)$). The second case cannot occur if we have a flow of order zero, as then the edge e may only have two vertices, say v, v', and then either $e = V(v)$ or $e = V(v')$, and the same argument applies. If we are to have a degenerate cycle, then this can only occur if both issues arise. These ideas are encapsulated in the following results.

Lemma 2.1 *Let V be a combinatorial vector field on an n-dimensional manifold $\mathbb{M}$. Then each 0-cell $\alpha \in \mathbb{M}$ is in at most one cycle.*

Proof: The proof considers cycles in the graph formed by the 2-skeleton of $\mathbb{M}$. If α is not in a cycle,

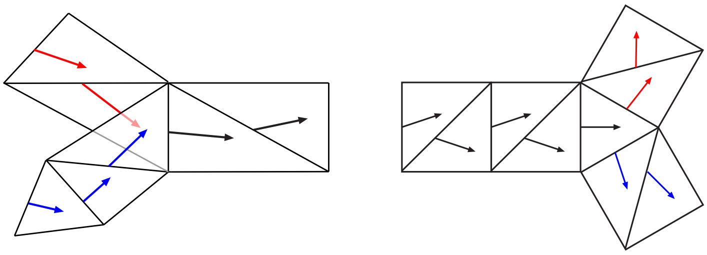

Figure 2.10: Two ways for a vector flow to be non-unique ('reverse' and 'forward' time).

then the lemma is true. So assume that α is in at least one cycle. Assume by way of contradiction that α is in two cycles, say $C_1 = \{\sigma_1, V(\sigma_1), \sigma_2, \ldots, V(\sigma_s)\}$ and $C_2 = \{\tau_1, V(\tau_1), \tau_2, \ldots, V(\tau_t)\}$. Clearly the two cycles must overlap, since α (and hence $V(\alpha)$) belong to both (by assumption). Cyclically permuting if necessary, arrange the cycles so that the first $2k$ elements of the cycles correspond, but no others. That is

$$\begin{aligned} \sigma_i &= \tau_i \qquad \forall\, i \leq k;\text{ and} \\ \sigma_i &\neq \tau_i \qquad \forall\, i > k. \end{aligned}$$

Now, since $V(\sigma_k)$ is a 1-cell, then either $C_1 = C_2 = \{\sigma_1, V(\sigma_1)\}$ (which can only occur if $\mathbb{M}$ is not quasiconvex) or it has two distinct vertices. Since σ_k must be one of these two, then $\sigma_{k+1} = \tau_{k+1}$, which is a contradiction and thus proves the lemma. $\square$

Using the negative of a vector field $-V$ (see Definition 1.20), we obtain the following corollary:

Corollary 2.2 *Let V be a combinatorial vector field on an n-dimensional manifold $\mathbb{M}$. Then each n-cell $\alpha \in \mathbb{M}$ is in at most one cycle.*

Proof: Take the inverse vector field $-V$ on $\mathbb{M}'$. Then each 0-cell $\alpha' \in \mathbb{M}'$ is in at most one cycle of $-V$. Since every cycle in a decomposition corresponds to one in the dual, then each n-cell $\alpha \in \mathbb{M}$ is in at most one cycle of V. $\square$

2.3 Euler Characteristic

Some common topological invariants are, by their very nature, combinatorial. The Euler characteristic is one such invariant, and is defined as:

$$\begin{aligned} \chi(\mathbb{M}) &= \sum_{e^i \in \mathbb{M}} (-1)^i \\ &= \sum_{p=0}^{n} (-1)^p \#\{p\text{-cells}\} \end{aligned}$$

where the e^i's are the cells of dimension i in a given decomposition of the manifold. Using combinatorial vector fields, we can derive the combinatorial analogue of the Poincaré-Hopf formula, which relates stationary points of vector fields to the euler characteristic:

Lemma 2.3

$$\chi(\mathbb{M}) = \sum_{p=0}^{n}(-1)^p\#\{\textit{stationary points of index } p\}$$

Proof: A combinatorial vector field consists of only stationary points and pairs of points $\{\sigma, V(\sigma)\}$ which satisfy $\dim(V(\sigma)) = \dim(\sigma)+1$. Thus we have that:

$$\begin{aligned}
\sum_{p=0}^{n}(-1)^p\#\{p\text{-cells}\} &= \sum_{p=0}^{n}(-1)^p\#\{\text{stationary points of index } p\} \\
&\quad + \sum_{p=0}^{n-1}(-1)^p\#\{\sigma \text{ of index } p\colon \exists\, V(\sigma)\} + \sum_{p=0}^{n-1}(-1)^{p+1}\#\{V(\sigma) \text{ of index } (p+1)\} \\
&= \sum_{p=0}^{n}(-1)^p\#\{\text{stationary points of index } p\} \\
&\quad + \sum_{p=0}^{n-1}(-1)^p\#\{\sigma \text{ of index } p : \exists\, V(\sigma)\} - \sum_{p=0}^{n-1}(-1)^p\#\{V(\sigma)\colon \sigma \text{ of index } p\} \\
&= \sum_{p=0}^{n}(-1)^p\#\{\text{stationary points of index } p\}.
\end{aligned}$$

□

This is illustrated by Figure 2.9, in which the Euler characteristic is

$$\begin{aligned}
\chi(S^3) &= \sum_{p=0}^{n}(-1)^p\#\{p\text{-cells}\} \\
&= \#\{0\text{-cells}\} - \#\{1\text{-cells}\} + \#\{2\text{-cells}\} - \#\{3\text{-cells}\} \\
&= 5 - 10 + 10 - 5 \\
&= 0.
\end{aligned}$$

And since all four 3-cells in the right-hand tetrahedron are rest points, as are the four blue 2-cells, then we have that

$$\begin{aligned}
\sum_{p=0}^{n}(-1)^p\#\{\text{stationary points of index } p\} &= 0 - 0 + 4 - 4 \\
&= 0 \\
&= \chi(S^3).
\end{aligned}$$

2.3.1 On the Global Gauss-Bonnet Theorem

Undoubtedly one of the more important results in smooth differential topology, the Gauss-Bonnet theorem links combinatorial invariants (namely the Euler characteristic) with Gaussian (mean) curvature. We first state the Theorem:

Theorem 2.4 (Global Gauss-Bonnet) *Let Σ be a closed surface. Then:*

$$\int_{\Sigma} \boldsymbol{\kappa}\, \mathrm{d}A = 2\pi\chi(\Sigma).$$

This result is quite powerful, however using Forman's definition of combinatorial curvature, it is easy to see that it cannot hold.

Lemma 2.5 *The Global Gauss-Bonnet Theorem does not hold for Forman's combinatorial Ricci curvature (see definition 1.9).*

Proof: Assume by way of contradiction that the Global Gauss-Bonnet theorem does hold up to a constant, say a. We shall consider the 2-sphere as a counter-example. Define $\mathbb{M}_{tetra}$ and $\mathbb{M}_{bucky}$ to be the tetrahedral and buckyball decompositions of the 2-sphere, respectively. Since we have a discrete space, then we need to sum in place of integration to obtain:

$$\begin{aligned}\sum_{e\in\mathbb{M}_{tetra}} \mathrm{Ric}(e) &= a.\chi(\mathbb{M}_{tetra}) \\ &= a.\chi(\mathbb{M}_{bucky}) \\ &= \sum_{e\in\mathbb{M}_{bucky}} \mathrm{Ric}(e).\end{aligned}$$

However, since:

$$\begin{aligned}\sum_{e\in\mathbb{M}_{bucky}} \mathrm{Ric}(e) &= \sum_{S^2} -c \\ &= -\frac{c}{4}\sum_{S^2} 4 \\ &= -\frac{c}{4}\sum_{e\in\mathbb{M}_{tetra}} \mathrm{Ric}(e)\end{aligned}$$

for some positive constant c, and thus

$$\sum_{\mathbb{M}_{tetra}} \kappa = -\frac{c}{4}\sum_{\mathbb{M}_{tetra}} \kappa$$

which gives a contradiction and proves the lemma. $\square$

2.4 Jacobi Fields

Jacobi fields are natural objects to examine. First, they are integral to the proof of Myers' Theorem in the smooth case (see, for example [19]); and second they exist only on the 2-skeleton of a complex, and hence may be helpful in the analysis of combinatorial curvature, which is only dependent upon the 2-skeleton (see Lemma 1.2).

In the smooth setting, Jacobi fields are a 1-parameter family of geodesics. As an example, consider a great semi-circle (from the north pole to the south pole) on the round 2-sphere. Then a Jacobi field for this geodesic is the set of paths which vary to the left and right. See Figure 2.11, where the original geodesic is in *red*, and a Jacobi field is shaded.

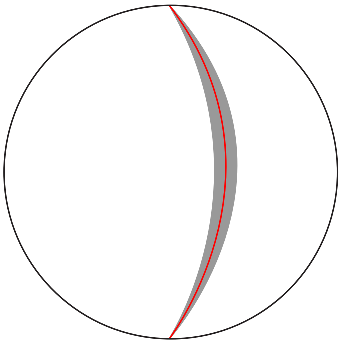

Figure 2.11: A smooth Jacobi field (shaded) near a geodesic (red) on the round S^2.

Combinatorially, we approximate Jacobi fields by taking a geodesic (shortest path) and generate an alternate path which may or may not be minimal. This is elaborated upon in Section 3.3.

A simple example of a Jacobi field is given in Figure 2.12. Here, the corresponding cells have been labeled $f_i = J(e_i)$, and shaded.

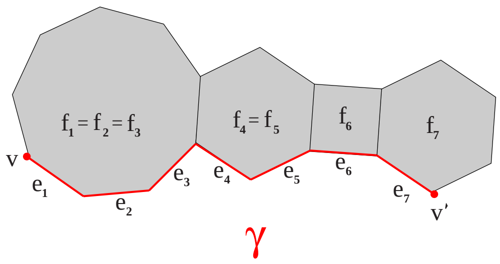

Figure 2.12: A Jacobi field along a path γ.

2.4.1 Continuation of Jacobi Fields

An important difference (which Forman notes in [7]) between combinatorial and Riemannian Jacobi fields is whether or not they can be extended from a segment to a whole path. In the Riemannian case, any Jacobi field over a segment γ' of a geodesic γ can always be extended to the entirety of γ. In the combinatorial setting, as they are defined here, this is not always possible. For example, consider Figure 2.13. Here, the field defined on the red sub-path γ' cannot be extended to all of γ. We now seek to make this notion precise.

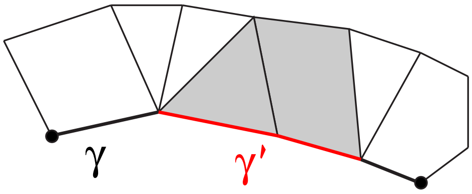

Figure 2.13: A Jacobi field on a subpath $\gamma' \subsetneq \gamma$ which cannot be extended to all of γ (bold).

Definition 2.1 (Continuation of Jacobi Fields) *Let* f_i *be a 2-cell with* $f_i \succ e_i$. *Then* f_i can be continued to e_{i+1} *if and only if there exists a 2-cell* $f_{i+1} \succ e_{i+1}$ *such that* f_i *and* f_{i+1} *share an edge other than* e_i *and* e_{i+1} *(*$f_i = f_{i+1}$ *is permitted).*

This is the same as saying that $J(e_i) = f_i$ and $J(e_{i+1}) = f_{i+1}$ forms a Jacobi field along the path $\hat{\gamma} = (v_{i-1}, e_i, v_i, e_{i+1}, v_{i+1})$ for some f_{i+1}, using the definition of Jacobi fields (see Definition 1.21). We shall return to this topic in more detail in Section 3.3.

Chapter 3:

Jacobi Fields and Myers' Theorem

We shall now state and prove our main result, namely Myers' Theorem. This gives a (finite) bound on the size of the fundamental group for any positively curved manifold, which then substantially limits the complexity of the topology which can be observed.

3.1 Myers' Theorem

Theorem 3.1 (Myers' Theorem) *Let* $\mathbb{M}$ *be a finite quasiconvex CW-complex, with a standard set of weights. If* $\mathrm{Ric}(e) > 0$ *for all edges* $e \in \mathbb{M}$*, then* $\pi_1(\mathbb{M})$ *is finite.*

We shall essentially follow Forman's proof of this result, as given in [7], with some elaboration and clarification. The general strategy is to find a finite bound for the diameter of the universal cover, which then ensures that the universal cover (and hence the fundamental group) are finite. Finding a bound for the diameter is quite involved, and will constitute the bulk of this chapter, in the proof of Theorem 3.5. Before attempting such an arduous endeavour, we begin with some simple, but crucial, lemmas; and culminate with the proof of Myers' Theorem.

3.2 Preliminaries to the Proof

Before we embark upon the proof of Myers theorem proper, we shall prove some elementary results which will be required. These are given with abbreviated proofs in [7].

Lemma 3.2 *Let* $\mathbb{M}$ *be a (not necessarily finite) connected CW-complex, where each vertex* v *satisfies* $\text{degree}(v) \leq \Delta$ *for some finite* Δ. *Then* $\mathbb{M}$ *has finitely many vertices if and only if* $\text{diam}(\mathbb{M})$ *is finite.*

Proof: Take some $v \in \mathbb{M}$. We now consider maximal paths beginning at v, which gives:

$$\begin{aligned} \#(\text{vertices in } \mathbb{M}) &\leq \sum_{\text{maximal paths } \mathcal{P}} \text{length}(\mathcal{P}) \\ &\leq V^{\max(\text{length}(\mathcal{P}))} \\ &\leq V^{\text{diam}(\mathbb{M})}. \end{aligned}$$

Since the number of vertices is bounded above by $V^{\text{diam}(\mathbb{M})}$ then if $\text{diam}(\mathbb{M})$ is finite, then so are the number of vertices. Otherwise, if the diameter is infinite, then by the definition of diameter, there is a path γ which has infinite length, and thus an infinite number of vertices. ☐

Lemma 3.3 *Let* $\mathbb{M}$ *be a quasiconvex CW-complex composed of cells* $\{\alpha\}$, *with a standard set of weights (see Definition 1.6). Let* $\widehat{\mathbb{M}}$ *be the same complex, composed of cells* $\{\hat{\alpha} = \alpha\}$, *where* $w_{\hat{\alpha}} = 1$ *for all* $\hat{\alpha}$. *Then*

$$\text{Ric}(e) = \omega_1 \text{Ric}(\hat{e}).$$

Proof: From Definition 1.15, we know that

$$\text{Ric}(e) = w_e \left(\left[\sum_{f^{(2)} \succ e} \frac{w_e}{w_f} + \sum_{v^{(0)} \prec e} \frac{w_v}{w_e} \right] - \sum_{e' \neq e} \left| \sum_{\substack{f^{(2)} \succ e \\ f^{(2)} \succ e'}} \frac{\sqrt{w_e w_{e'}}}{w_f} - \sum_{\substack{v^{(0)} \prec e \\ v^{(0)} \prec e'}} \frac{w_v}{\sqrt{w_e w_{e'}}} \right| \right).$$

So using our standard set of weights we obtain

$$\begin{aligned} \text{Ric}(e) &= \omega_1\omega_2 \left(\left[\sum_{f^{(2)} \succ e} \frac{\omega_1\omega_2}{\omega_1\omega_2^2} + \sum_{v^{(0)} \prec e} \frac{\omega_1}{\omega_1\omega_2} \right] - \sum_{e' \neq e} \left| \sum_{\substack{f^{(2)} \succ e \\ f^{(2)} \succ e'}} \frac{\sqrt{(\omega_1\omega_2)(\omega_1\omega_2)}}{\omega_1\omega_2^2} - \sum_{\substack{v^{(0)} \prec e \\ v^{(0)} \prec e'}} \frac{\omega_1}{\sqrt{(\omega_1\omega_2)(\omega_1\omega_2)}} \right| \right) \\ &= \omega_1\omega_2 \left(\left[\sum_{f^{(2)} \succ e} \frac{1}{\omega_2} + \sum_{v^{(0)} \prec e} \frac{1}{\omega_2} \right] - \sum_{e' \neq e} \left| \sum_{\substack{f^{(2)} \succ e \\ f^{(2)} \succ e'}} \frac{1}{\omega_2} - \sum_{\substack{v^{(0)} \prec e \\ v^{(0)} \prec e'}} \frac{1}{\omega_2} \right| \right) \\ &= \omega_1 \left(\#\left\{ f^{(2)} \succ e \right\} + \#\left\{ v^{(0)} \prec e \right\} - \sum_{e' \neq e} \left| \#\left\{ f^{(2)} \succ e : f^{(2)} \succ e' \right\} - \#\left\{ v^{(0)} \prec e : v^{(0)} \prec e' \right\} \right| \right). \end{aligned}$$

We now consider the final sum, treating each of four (collectively exhaustive) cases separately:

(i) e' is a 0-neighbour of e;

(ii) e' is a 2-neighbour of e;

(iii) e' is transverse to e; and

(iv) e' is distant from e.

Case (i): If e' is a 0-neighbour of e, then there is precisely one v such that $v \prec e$ and $v \prec e'$. If there was another, say $v' \neq v$, which was also shared then this would imply that $e = e'$ by quasiconvexity. Also, since e' is a 0-neighbour of e, there are no 2-cells f which satisfy both $f \succ e$ and $f \succ e'$. So the term in the sum is:

$$\begin{aligned} \left|\#\left\{f^{(2)} \succ e : f^{(2)} \succ e'\right\} - \#\left\{v^{(0)} \prec e : v^{(0)} \prec e'\right\}\right| &= |0-1| \\ &= 1. \end{aligned}$$

Case (ii): If e' is a 2-neighbour of e, then there is precisely one f such that $f \succ e$ and $f \succ e'$, again by quasiconvexity. Since e' is a 2-neighbour of e, there are no 0-cells v which satisfy both $v \prec e$ and $v \prec e'$. So the term in the sum is:

$$\begin{aligned} \left|\#\left\{f^{(2)} \succ e : f^{(2)} \succ e'\right\} - \#\left\{v^{(0)} \prec e : v^{(0)} \prec e'\right\}\right| &= |1-0| \\ &= 1. \end{aligned}$$

Case (iii): If e and e' are transverse, then we know that there is at least one 0-cell v and at least one 2-cell f such that $v \prec e \prec f$ and $v \prec e' \prec f$. Once again, quasiconvexity ensures that there is precisely one of each. Examining the term in the sum, we get

$$\begin{aligned} \left|\#\left\{f^{(2)} \succ e : f^{(2)} \succ e'\right\} - \#\left\{v^{(0)} \prec e : v^{(0)} \prec e'\right\}\right| &= |1-1| \\ &= 0. \end{aligned}$$

Case (iv): Finally, if e and e' are distant, then

$$\begin{aligned} \left|\#\left\{f^{(2)} \succ e : f^{(2)} \succ e'\right\} - \#\left\{v^{(0)} \prec e : v^{(0)} \prec e'\right\}\right| &= |0-0| \\ &= 0. \end{aligned}$$

Since only 0- and 2-neighbours contribute (and indeed, are the only contributors) to the sum, then we can rewrite the curvature as

$$\begin{aligned} \mathrm{Ric}(e) &= \omega_1 \left(\#\left\{f^{(2)} \succ e\right\} + \#\left\{v^{(0)} \prec e\right\} - \sum_{e' \neq e} \left|\#\left\{f^{(2)} \succ e : f^{(2)} \succ e'\right\} - \#\left\{v^{(0)} \prec e : v^{(0)} \prec e'\right\}\right|\right) \\ &= \omega_1 \left(\#\left\{f^{(2)} \succ \hat{e}\right\} + \#\left\{v^{(0)} \prec \hat{e}\right\} - [\#\{0-\text{neighbours}\} + \#\{2-\text{neighbours}\}]\right) \\ &= \omega_1 \,\mathrm{Ric}(\hat{e}), \end{aligned}$$

which proves the lemma. □

We now turn our attention to a tool which is invaluable in proving Myers' theorem, namely Jacobi fields. Forman uses combinatorial Jacobi fields in [7] to find a restriction upon the number of distinctly different (in some well-defined sense) ways in which a geodesic path can be deformed whilst maintaining its length. This is then used to obtain a bound for the diameter of the covering space. This is analogous to the smooth (Riemannian) setting, where Jacobi fields are used to deform geodesics in orthogonal directions, and again this ultimately leads to an upper bound. For definitions and examples, see Section 2.4. Here we shall prove an elementary result and then proceed to a major theorem.

Lemma 3.4 (Uniqueness) *Let $\gamma = v_0, e_1, \cdots, e_k, v_k$ be a connected path. If J_1 and J_2 are combinatorial Jacobi fields along γ and*

$$J_1(e_i) = J_2(e_i)$$

for some i. Then for $1 \leq j \leq k$,

$$J_1(e_j) = J_2(e_j).$$

Proof: We shall use induction, proving two cases:

- **(i)** $J_1(e_{i+1}) = J_2(e_{i+1})$ for $i < k$; and
- **(ii)** $J_1(e_{i-1}) = J_2(e_{i-1})$ for $i > 1$.

Case (i): If $J_1(e_i) = J_1(e_{i+1})$ then both $e_i \prec J_1(e_i)$ and $e_{i+1} \prec J_1(e_i)$. By definition, we also know that $J_2(e_{i+1})$ shares an edge, say e', with $J_2(e_i) = J_1(e_i)$ other than e_i and e_{i+1}. Then e' and e_{i+1} are both edges of $J_2(e_{i+1})$ and $J_2(e_i)$ and so by quasiconvexity we have:

$$J_2(e_{i+1}) = J_2(e_i) = J_1(e_i) = J_1(e_{i+1}).$$

If $J_1(e_i) \neq J_1(e_{i+1})$ then $J_1(e_i)$ and $J_1(e_{i+1})$ share an edge, e' say. By quasiconvexity, we know that:

$$\bar{J_1(e_i)} \cap \bar{J_1(e_{i+1})} = \bar{e'}$$

and furthermore, since $J_1(e_i) \succ e_i \succ v_i$ and $J_1(e_{i+1}) \succ e_{i+1} \succ v_i$ then

$$v_i \in \bar{J_1(e_i)} \cap \bar{J_1(e_{i+1})}$$

and hence $e' \succ v_i$.

Consider J_2. If $J_2(e_i) = J_2(e_{i+1})$, then by the earlier argument, $J_1(e_i) = J_1(e_{i+1})$, which is a contradiction. So $J_2(e_i) \neq J_2(e_{i+1})$, and then $J_2(e_i)$ and $J_2(e_{i+1})$ share an edge, say e''. Using the same argument as for J_1, we obtain that $e'' \succ v_i$ where $J_2(e_i) \succ e''$ and $J_2(e_{i+1}) \succ e''$ with $e'' \notin \{e_i, e_{i+1}\}$.

Now, $J_1(e_i) = J_2(e_i)$ is a 2-cell, and hence has exactly two edges which have v_i as an endpoint. We know that e_i is one such edge. Both e' and e'' are edges distinct from e_i which satisfy these two properties, and hence

$$e' = e''.$$

Then $J_1(e_{i+1})$ and $J_2(e_{i+1})$ both share the edges e' and e_{i+1}, and so by quasiconvexity we can conclude that

$$J_1(e_{i+1}) = J_2(e_{i+1}).$$

Case (ii): Assume $J_1(e_{i-1}) = J_1(e_i)$. By definition, there exists an edge $e' \prec J_2(e_{i-1})$ such that $e' \prec J_2(e_i) = J_1(e_i)$. Then e' and e_{i-1} are both edges of $J_2(e_{i-1})$ and $J_2(e_i)$ and so by quasiconvexity we have:

$$J_2(e_{i-1}) = J_2(e_i) = J_1(e_i) = J_1(e_{i-1}).$$

Now assume that $J_1(e_{i-1}) \neq J_1(e_i)$. Again, $J_2(e_i) = J_2(e_{i-1})$ implies by the earlier argument that $J_1(e_i) = J_1(e_{i-1})$, which is a contradiction. So $J_2(e_i) \neq J_2(e_{i-1})$, and using quasiconvexity we again obtain that:

$$\exists\, e' \neq e_i : \; e' \prec J_1(e_i) = J_2(e_i) \text{ and } J_1(e_{i-1}) \succ e' \succ v_{i-1}.$$

Likewise, for J_2 we find

$$\exists\, e'' \neq e_i : \; e'' \prec J_1(e_i) = J_2(e_i) \text{ and } J_1(e_{i-1}) \succ e'' \succ v_{i-1}.$$

Since $J_1(e_i) = J_2(e_i)$ is a 2-cell, it has exactly two edges which have v_i as an endpoint. We know that e_i is one such edge. Both e' and e'' are edges distinct from e_i which satisfy these two properties, and hence

$$e' = e''.$$

Then $J_1(e_{i-1})$ and $J_2(e_{i-1})$ both share the edges e' and e_{i-1}, and so by quasiconvexity we can conclude that

$$J_1(e_{i-1}) = J_2(e_{i-1}).$$

Thus by induction we can conclude that if $J_1(e_i) = J_2(e_i)$ for some i, then for all $j \in [k]$, $J_1(e_j) = J_2(e_j)$, which proves the lemma. $\square$

3.3 Bounds for the Diameter of $\mathbb{M}$, Using Jacobi Fields

We now seek to use combinatorial Jacobi fields to show that there is a bound on the diameter of a quasiconvex CW-complex $\mathbb{M}$. We will then apply this to our universal cover to deduce Myers' Theorem in Section 3.4. Before we state (and prove) this result we will define some notation for convenience. Recall that Jacobi fields on a segment of a path cannot always be extended to the entirety of that path (see Definition 2.1). Then let

$$C(e_i, e_{i+1}) = \{f^{(2)} \succ e_i \text{ such that } f \text{ can be continued to } e_{i+1}\}$$

and

$$NC(e_i, e_{i+1}) = \{f^{(2)} \succ e_i \text{ such that } f \text{ cannot be continued to } e_{i+1}\}.$$

Also define $C(e_i, e_{i-1})$ and $NC(e_i, e_{i-1})$ analogously.

We now state the theorem:

Theorem 3.5 *Let $\mathbb{M}$ be a (not necessarily finite) quasiconvex CW complex, with the cells assigned a standard set of weights $\omega_\alpha = \omega_1 \cdot \omega_2^{(\dim \alpha)}$. Define*

$$T(\mathbb{M}) = \left[\sup_{e \in \mathbb{M}} \#\{f^{(2)} \succ e\} \right] + 1,$$

where the supremum is over all edges $e \in \mathbb{M}$. If $\mathrm{Ric}(e) \geq c > 0$ for all edges e, then

$$\mathrm{diam}(\mathbb{M}) \leq \frac{2\omega_1}{c} T(\mathbb{M}).$$

Proof: By Lemma 3.3, we know that changing the weight on each cell to 1 effectively divides the Ricci curvature by ω_1. Therefore we prove the theorem in the case that all cells have been assigned the weight 1. Let v and v' be two vertices of $\mathbb{M}$, and let

$$\gamma: \ v = v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_k = v'$$

be a minimal path from v to v'. Namely, distance$(v, v') = k$. Now, for every $1 \leq i \leq k$,

$$c \leq \mathrm{Ric}(e_i) = \#\{f^{(2)} \succ e_i\} + 2 - \#\{0\text{-neighbours of } e_i\} - \#\{2\text{-neighbours of } e_i\}.$$

We first examine the 2-neighbours of e_i. Take any 2-cell $f \succ e_i$ with [sides(f)] 1-cells in the boundary of f. Then

$$\begin{aligned}
\#\{2\text{-neighbours from } f\} &= \mathrm{sides}(f) - \#\{e' \prec f : \bar{e'} \cap \bar{e}_i \neq \emptyset\} \\
&= \mathrm{sides}(f) - \#\{e_i, e', e''\} \\
&= \mathrm{sides}(f) - 3,
\end{aligned}$$

where e' and e'' are the two edges of f which share an endpoint with e_i. Since every 2-neighbour arises in this way, then

$$\#\{\text{2-neighbours of } e_i\} = \sum_{f^{(2)} \succ e_i} (\text{sides}(f) - 3)$$

which gives

$$\begin{aligned}\#\{f^{(2)} \succ e_i\} - \#\{\text{2-neighbours of } e_i\} &= \sum_{f^{(2)} \succ e_i} 1 + \sum_{f^{(2)} \succ e_i} (3 - \text{sides}(f)) \\ &= \sum_{f^{(2)} \succ e_i} (4 - \text{sides}(f)).\end{aligned}$$

We now consider the 0-neighbours of e_i. Assume first that $i > 1$. If $f \succ e_{i-1}$ is a 2-cell, then there is exactly one edge e' such that $f \succ e' \succ v_{i-1}$. Now, e' is a 0-neighbour if and only if there does not exist an $f' \succ e'$ which also has $f' \succ e_i$. This is true if and only if $f \in NC(e_{i-1}, e_i)$.

Similarly, if we assume that $i < k$, then we get precisely one 0-neighbour of e_i from $NC(e_{i+1}, e_i)$. Also, we have that e_{i-1} is a 0-neighbour of e_i if and only if e_i and e_{i-1} do not share a 2-cell, and likewise e_{i+1} is a 0-neighbour of e_i unless e_i and e_{i+1} share a 2-cell. Let $\epsilon_i \in \{0, 1, 2\}$ be the number of 1-cells from the set $\{e_{i-1}, e_{i+1}\}$ which share a face with e_i. Thus for $2 \leq i \leq k - 1$:

$$\#\{\text{0-neighbours of } e_i\} \geq NC(e_{i-1}, e_i) + NC(e_{i+1}, e_i) + (2 - \epsilon_i),$$

and so

$$\text{Ric}(e_i) \leq \sum_{f^{(2)} \succ e_i} (4 - \text{sides}(f)) - NC(e_{i-1}, e_i) - NC(e_{i+1}, e_i) + \epsilon_i.$$

We now apply the same argument to $i = 1$, with the proviso that $\epsilon_1 = 1$ if e_1 and e_2 share a 2-cell, and zero otherwise. Since we need only consider edges of the type e_{i+1} (further along the path) then

$$\begin{aligned}\text{Ric}(e_1) &\leq \sum_{f^{(2)} \succ e_1} (4 - \text{sides}(f)) + 2 - NC(e_2, e_1) + (\epsilon_1 - 1) \\ &= \sum_{f^{(2)} \succ e_1} (4 - \text{sides}(f)) + 1 - NC(e_2, e_1) + \epsilon_1,\end{aligned}$$

and if $i = k$ then we need only consider the edges of the type e_{i-1} (earlier along the path) and

$$\text{Ric}(e_k) \leq \sum_{f^{(2)} \succ e_k} (4 - \text{sides}(f)) + 1 - NC(e_{k-1}, e_k) + \epsilon_k.$$

Using the property that $\text{Ric}(e_i) \geq c$ for all i, and then summing, we obtain

$$ck \leq 2 + \sum_{i=1}^{k} \sum_{f^{(2)} \succ e_i} (4 - \text{sides}(f)) - \sum_{i=1}^{k-1} NC(e_{i+1}, e_i) - \sum_{i=2}^{k} NC(e_{i-1}, e_i) + 2\epsilon(\gamma), \qquad (3.1)$$

where $\epsilon(\gamma)$ is the number of pairs $(e_i, e_{i+1}) \in \gamma$ which share a face.

We will now use a Jacobi field to produce an alternate path which is at least as long as γ. Consider the subpath $\gamma' = v_r, e_{r+1}, \ldots, v_{s-1}, e_s, v_s$ of γ. Since γ is a minimal path, then so too is γ'. Let J be a Jacobi field along γ'. Then J is a series of connected 2-cells adjacent to γ', and schematically there is thus a bigon with interior J, vertices v_r, v_s and edges γ' and $\tilde{\gamma}$. The original path γ is in **bold**. This is illustrated in Figure 3.1.

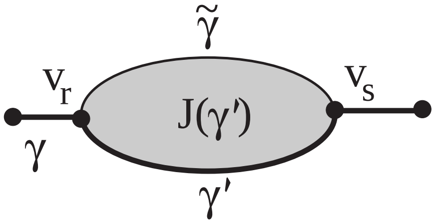

Figure 3.1: Alternative path from a Jacobi field (schematic).

We shall now clarify the exact definition of the alternate path $\tilde{\gamma}$ with particular reference to Figure 3.2, noting that we have written $f_i = J(e_i)$, and that γ' is *red*, $\tilde{\gamma}$ is *blue*.

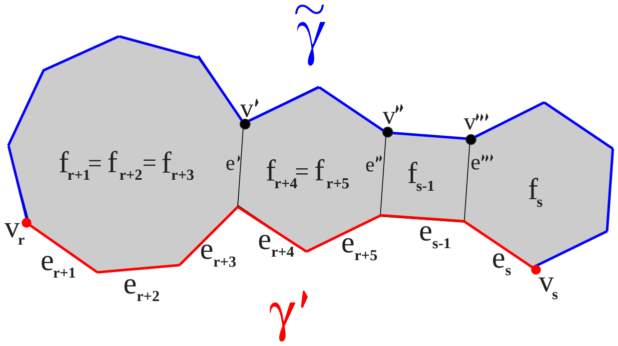

Figure 3.2: An alternative path from a Jacobi field (explicit).

Suppose that $J(e_{r+1}) = J(e_{r+2}) = \cdots = J(e_{r+t-1}) \neq J(e_{r+t})$. Then the first $t-1$ edges of γ' all share a 2-cell, namely $J(e_{r+1})$. Since γ is a minimal path, then so too is any section of it, and in particular γ' is minimal. This ensures that all of the edges $e_{r+1}, e_{r+2}, \ldots, e_{r+t-1}$ are distinct. Now consider $J(e_{r+t})$. By the definition of a Jacobi field, we know that $J(e_{r+t})$ shares an edge with $J(e_{r+1})$, say e'. Additionally, we know that v_{r+t-1} is an endpoint of e'. Let v' be the other endpoint of e'. Begin at v_r and head around the boundary of $J(e_{r+1})$ away from e_{r+1}, until you reach v'.

This gives a path of length $\text{sides}(J(e_{r+1})) - [(t-1)+1] = \text{sides}(J(e_{r+1})) - t$, since we exclude the edges in γ' and also e'. Now suppose that $J(e_{r+t}) = J(e_{r+t+1}) = \cdots = J(e_{r+t+u-1}) \neq J(e_{r+t+u})$. Then as before, we know that the next u edges of γ' all share the 2-cell $J(e_{r+t})$, and are distinct. Also, $J(e_{r+t})$ and $J(e_{r+t+u})$ share an edge, say e'' with endpoints $v_{r+t+u-1}$ and v''. We now start at v' and walk around the boundary of $J(e_{r+t})$ (away from e') until we get to v''. This path has length $\text{sides}(J(e_{r+1})) - [(u)+1+1] = \text{sides}(J(e_{r+1})) - (u+2)$ since we do not walk along edges in γ' nor the edges e, e'. If we continue doing this, then we obtain an alternate path $\tilde{\gamma}$ from v_r to v_s. In Figure 3.2, this is illustrated for the case $t = 3$, $u = 2$.

Let us now introduce some notation. We shall denote a maximal connected set along which J is constant by $U \subset \{e_{r+1}, \ldots, e_s\}$. Let $J(U)$ be the 2-cell where $J(U) = J(e_i)$ for all $e_i \in U$, and $\#U$ be the number of elements in U. For a U in the interior of the path $\tilde{\gamma}$, that is $v_r, v_s \notin U$, then the alternate path (as described above) will have

$$\text{length}(\tilde{\gamma}|_U) = \text{sides}(J(U)) - (\#U + 2)$$

while a boundary U (one containing either v_r or v_s) will satisfy

$$\text{length}(\tilde{\gamma}|_U) = \text{sides}(J(U)) - \#U - \#(U \cap \{v_r, v_s\})$$

and so the total length of the alternate path is

$$\text{length}(\tilde{\gamma}) = 2 + \sum_U [\text{sides}(J(U)) - (\#U + 2)], \tag{3.2}$$

which is obtained by summing over the maximal connected subsets U. Since γ' is a minimal path, then every summand in equation 3.2 is non-negative, and so

$$\begin{aligned}
\text{length}(\tilde{\gamma}) &\leq 2 + \sum_U \#U[\text{sides}(J(U)) - (\#U + 2)] \\
&= 2 + \sum_U \#U[\text{sides}(J(U)) - 3] - \sum_U \#U(\#U - 1) \\
&= 2 + \sum_{i=r+1}^{s} [\text{sides}(J(e_i)) - 3] - \sum_U \#U(\#U - 1) \\
&\leq 2 + \sum_{i=r+1}^{s} [\text{sides}(J(e_i)) - 3] - 2\sum_U (\#U - 1).
\end{aligned}$$

Now, let $\delta(J) = \#\{(i, i+1) : J(e_i) = J(e_{i+1})\}$. Then each set U contributes $\#U - 1$ such pairs, and so

$$\delta(J) = \sum_U (\#U - 1)$$

which gives

$$\text{length}(\tilde{\gamma}) \leq 2 + \sum_{i=r+1}^{s} [\text{sides}(J(e_i)) - 3] - 2\delta(J).$$

Now, since γ' is minimal, then we know that

$$\text{length}(\tilde{\gamma}) \geq \text{length}(\gamma') = s - r = \sum_{i=r+1}^{s} 1,$$

and so

$$0 \leq 2 + \sum_{i=r+1}^{s} [\text{sides}(J(e_i)) - 4] - 2\delta(J). \tag{3.3}$$

Now, we define that a Jacobi field J is *maximal* if it cannot be extended along a larger subpath of γ. That is, J is maximal if there is no connected path γ'' such that $\gamma' \subsetneq \gamma'' \subset \gamma$ such that J can be extended to γ''. From Lemma 3.4, we know that for any edge $e_i \in \gamma$ and for all 2-cells $f \succ e_i$ that $J(e_i) = f$ in at most one maximal Jacobi field. Trivially, $J(e_i) = f$ forms a Jacobi field over the path $\gamma' = v_{i-1}, e_i, v_i$ and so there is exactly one such maximal Jacobi field. We now sum equation 3.3 over maximal Jacobi fields to obtain

$$0 \leq 2\#\{\text{maximal Jacobi fields}\} + \sum_{i=r+1}^{s} \sum_{f^{(2)} \prec e_i} [\text{sides}(f) - 4] - 2 \sum_{\substack{\text{maximal Jacobi} \\ \text{fields } J}} \delta(J). \tag{3.4}$$

We turn our attention to the last term, and consider an edge e_i, and two Jacobi fields $J_1 \neq J_2$ on the subpath $\gamma' = v_{i-1}, e_i, v_i, e_{i+1}, v_{i+1}$. If $\delta(J_1) = \delta(J_2) = 0$, then there is no contribution to the last term of equation 3.4. Without loss of generality (relabel if necessary), assume that $J_1(e_i) = J_1(e_{i+1})$. Then e_i and e_{i+1} share a face, and contribute to $\epsilon(\gamma')$. If both $J_1(e_i) = J_1(e_{i+1})$ and $J_2(e_i) = J_2(e_{i+1})$, then we have two 2-cells with two common edges in their boundary, which contradicts quasiconvexity. Thus

$$\sum_{\substack{\text{maximal Jacobi} \\ \text{fields } J}} \delta(J) = \epsilon(\gamma).$$

If we now substitute this and equation 3.4 into equation 3.1, then

$$ck \leq 2 + 2\#\{\text{maximal Jacobi fields}\} - \sum_{i=1}^{k-1} NC(e_{i+1}, e_i) - \sum_{i=2}^{k} NC(e_{i-1}, e_i). \tag{3.5}$$

Now, a Jacobi field on a subpath $\gamma' = v_r, \ldots, v_s$ is maximal if and only if

(i) $r = 0$ or $J(e_{r+1}) \in NC(e_{r+1}, e_r)$; and
(ii) $s = k$ or $J(e_s) \in NC(e_s, e_{s+1})$.

If we now map each maximal Jacobi field to its initial endpoint, $J(e_{r+1})$, then we have a bijection between $\{\text{maximal Jacobi fields}\}$ and $\{f^{(2)} \succ e_1\} \cup \{NC(e_i, e_{i-1})\}_{i=2}^{k}$. Thus we can conclude that

$$\#\{\text{maximal Jacobi fields}\} = \#\{f^{(2)} \succ e_1\} + \sum_{i=2}^{k} \{NC(e_i, e_{i-1})\}.$$

Alternatively, we could map each Jacobi field to its final endpoint, and obtain a bijection which gives

$$\#\{\text{maximal Jacobi fields}\} = \#\{f^{(2)} \succ e_k\} + \sum_{i=1}^{k-1}\{NC(e_i, e_{i+1})\}.$$

If we substitute these into equation 3.5, then

$$ck \leq 2 + \#\{f^{(2)} \succ e_1\} + \#\{f^{(2)} \succ e_k\} \leq 2T(\mathbb{M}),$$

and so

$$\text{distance}(v, v') = k \leq \frac{2}{c}T(\mathbb{M}).$$

Since v and v' were arbitrary points, then this bound applies for all possible pairs, and hence

$$\text{diam}(\mathbb{M}) \leq \frac{2}{c}T(\mathbb{M}).$$

□

It is possible to relax the assumptions for Theorem 3.5 slightly, if we have the condition that paths are *uniformly curved.* This is neither stated nor proved in Forman's paper [7], and is original work:

Definition 3.1 *Let $\mathbb{M}$ be a quasiconvex CW-complex, such that every edge $e \in \mathbb{M}$ has Ric$(e) \geq 0$. Let $C_{\min}$ be the smallest cycle in $\mathbb{M}$ which does not bound a 2-cell. Then if there exists a $B < |C_{\min}|$ such that for any path γ with $|\gamma| \geq B$ we have that*

$$\sum_{e\in\gamma}\text{Ric}(e) \geq 1,$$

then we say that $\mathbb{M}$ is uniformly curved *with bound B.*

Corollary 3.6 *Let $\mathbb{M}$ be a (not necessarily finite) quasiconvex CW complex, with the cells assigned a standard set of weights $\omega_\alpha = \omega_1 \cdot \omega_2^{(\dim\alpha)}$. Define*

$$T(\mathbb{M}) = \left\lceil \sup_{e\in\mathbb{M}} \#\{f^{(2)} \succ e\} \right\rceil + 1,$$

where the supremum is over all edges $e \in \mathbb{M}$. If $\mathbb{M}$ is uniformly curved with bound B, then

$$\text{diam}(\mathbb{M}) \leq \frac{2.B.\omega_1}{c}T(\mathbb{M}).$$

Proof: Take a geodesic path of length k. Then it consists of k/B subpaths, each of which must have total Ricci curvature at least one, and so

$$\sum_{e\in\gamma}\text{Ric}(e) \geq \frac{k}{B}.$$

Then we can apply the proof of Theorem 3.5, with the proviso that when we sum over curvature (see Equation 3.1), we obtain

$$\frac{ck}{B} \leq 2 + \sum_{i=1}^{k} \sum_{f^{(2)} \succ e_i} (4 - \text{sides}(f)) - \sum_{i=1}^{k-1} NC(e_{i+1}, e_i) - \sum_{i=2}^{k} NC(e_{i-1}, e_i) + 2\epsilon(\gamma),$$

which yields the desired result. □

3.4 Proof of Myers' Theorem

We are now in a position to prove our main result, namely Myers' Theorem. We first re-state the theorem:

Theorem 3.1 *Let* $\mathbb{M}$ *be a finite quasiconvex CW-complex, with a standard set of weights. If* $\text{Ric}(e) > 0$ *for all edges* $e \in \mathbb{M}$*, then* $\pi_1(\mathbb{M})$ *is finite.*

Proof: Let $\mathbb{M}$ be a finite CW-complex, where cells have been assigned a standard set of weights. If $\text{Ric}(e) > 0$, then since $\mathbb{M}$ is finite, there exists some $c > 0$ such that $\text{Ric}(e) \geq c$ for all edges e. Again using the property that $\mathbb{M}$ is finite, we know that for any vertex v, $\text{degree}(v) \leq \#\{\text{vertices}\} = V$, and also that $T(\mathbb{M})$ is finite.

Now consider $\pi : \tilde{\mathbb{M}} \to \mathbb{M}$, the universal cover of $\mathbb{M}$, using the induced cell structure from $\mathbb{M}$. The induced structure on $\tilde{\mathbb{M}}$ is that if $\tilde{\alpha}$ is a p-cell of $\tilde{\mathbb{M}}$, then π maps $\tilde{\alpha}$ homeomorphically onto a p-cell of $\mathbb{M}$, along with appropriate weights. Since locally these two descriptions are identical (this is assured by quasiconvexity), then for all edges $\tilde{e}$ in $\tilde{\mathbb{M}}$,

$$\text{Ric}_{\tilde{\mathbb{M}}}(\tilde{e}) = \text{Ric}_{\mathbb{M}}(\pi(\tilde{e})).$$

In particular, we know that $\text{Ric}_{\tilde{\mathbb{M}}}(\tilde{e}) \geq c$ for all edges $\tilde{e}$ in $\tilde{\mathbb{M}}$. Again since there is a local correspondence, for every edge $\tilde{e}$ of $\tilde{\mathbb{M}}$,

$$\#\{\tilde{f}^{(2)} \succ \tilde{e}\}_{\tilde{\mathbb{M}}} = \#\{f^{(2)} \succ \pi(\tilde{e})\}_{\mathbb{M}},$$

and so

$$T(\tilde{\mathbb{M}}) = T(\mathbb{M}) < \infty.$$

For each vertex $\tilde{v}$ of $\tilde{\mathbb{M}}$,

$$\text{degree}(\tilde{\mathbb{M}}) = \text{degree}(\pi(\tilde{\mathbb{M}})) \leq V.$$

We can now apply Theorem 3.5, and find that the diameter of $\tilde{\mathbb{M}}$ is finite. If we let $\Delta = V$, then Lemma 3.2 applies, and so $\tilde{\mathbb{M}}$ has finitely many vertices. Now

$$\#\pi_1(\mathbb{M}) = \frac{\#\{\text{vertices of } \tilde{\mathbb{M}}\}}{\#\{\text{vertices of } \mathbb{M}\}}$$

which implies that $\pi_1(\mathbb{M})$ is finite. □

Chapter 4:

An Alternative Approach to Combinatorial Curvature

This chapter comprises three parts. First, an examination of the limitations of Forman's theory is undertaken, including attempts to quantify its shortcomings. Second, an alternative method of calculating combinatorial curvature, here called *geometric curvature*, is proposed. Some results which are possible from this definition, including the Gauss-Bonnet Theorem (see Theorem 2.4) are proved. We conclude with a discussion on possible methods with which to prove some results which any sensible formulation of combinatorial curvature ought to satisfy. All proofs in this chapter are original work of the author.

4.1 Limitations of Forman's Combinatorial Curvature

Any definition of combinatorial curvature must, as a minimum requirement, satisfy analogues to key results in the smooth theory. Two of the most important results are the Gauss-Bonnet Theorem (see Theorem 2.4) and Myers' Theorem (see Theorem 3.1). As we have shown, Forman's definition of combinatorial curvature admits a proof of Myers' Theorem, while simultaneously denying Gauss-Bonnet (see Lemma 2.5). This is the key weakness of his formulation, and raises the question of whether this method is appropriate.

As was discussed in section 2.1, there is also a paucity of examples which satisfy Forman's two conditions, namely quasiconvexity and positive Ricci curvature. The main reason for this is because as a particular cell-decomposition is subdivided to create one which is quasiconvex (see Lemma 1.3), the Ricci curvature becomes smaller at some points, and is eventually negative. Clearly a theory which is characterised by its calculability is severely hampered if the conditions are too strong to be satisfied by many interesting examples, as it then can rarely be applied to general CW-complexes.

In an effort to rectify this, we were able to slightly relax the assumptions used to prove the main result, Myers' Theorem. First, it was shown that only the 2-skeleton needed to be quasiconvex (see Lemma 1.2). Second, the restriction that the curvature needed to be positive for every edge was able to be relaxed to an insistence that the manifold is uniformly curved (see Corollary 3.6). Ultimately, these improvements are small and unable to overcome the difficulties with Forman's conception of combinatorial Ricci curvature.

4.2 Geometric Curvature

The aim here is to propose an alternative formulation of combinatorial curvature, which both allows more examples to be generated, and also admits results which were excluded from Forman's theory. One key result which any sensible definition of combinatorial curvature should permit is the Gauss-Bonnet theorem (see Theorem 2.4), however Forman's formulation denies this result (see Lemma 2.5). The more geometric version of combinatorial curvature, developed by the author in conjunction with Hyam Rubinstein and presented here, satisfies the Gauss-Bonnet theorem for surfaces (see Theorem 4.2). It is not known if this conception of curvature will allow a proof of Myers' Theorem, but possibilities are discussed in Section 4.4.

4.2.1 Geometric Curvature of Surfaces

We now seek to motivate our definition of combinatorial curvature, here called *geometric curvature*. Classically, curvature developed from analysis of curves and surfaces embedded in 3-dimensional euclidean space. One method is to examine geodesic triangles, and determine whether or not the angle sum is more or less than expected. This easily generalises to polygons, and we can formulate the following definition:

Definition 4.1 *Let P be a polygon in a 2-dimensional closed manifold $\mathbb{M}$. Then the* geometric curvature *κ of P is*

$$\kappa(P) = \sum_{v \in P} (\textit{angle at } v \textit{ in } P) - \pi(\text{sides}(P) - 2).$$

This encapsulates the idea that on positively curved surfaces (such as the round sphere) the angle sum of a geodesic triangle is greater than one in the Euclidean plane. This is illustrated in Figure 4.1.

The most general way to calculate angles at vertices is to obtain the single constraint that the angles at a vertex must sum to 2π, and then solve a system of simultaneous equations to determine the angles required for the curvature to be positive (or other desired condition). However, such a complicated procedure greatly increases the complexity (in computational terms) and undermines the key advantage of a combinatorial approach: its calculability (as discussed in Chapter 1).

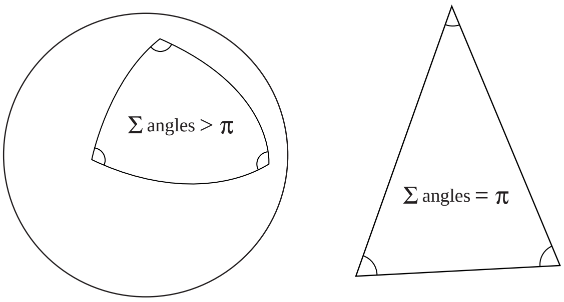

Figure 4.1: Geodesic triangles on the round sphere and in the Euclidean plane.

Instead, we can approximate by allocating the angles equally at a vertex. That is, for any vertex v, any angle at that vertex is

$$\text{angle at } v = \frac{2\pi}{d(v)}.$$

This is illustrated in Figure 4.2.

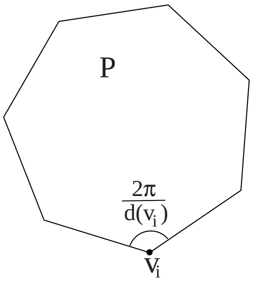

Figure 4.2: Combinatorial angle of a polygon (approximated).

Then we can calculate the curvature of some arbitrary polygon P as

$$\begin{aligned}\kappa(P) &= \sum_{v_i \in P} (\text{angle at } v_i \text{ in } P) - \pi(\text{sides}(P) - 2) \\ &= \sum_{v_i \prec P} \frac{2\pi}{d(v_i)} - \pi(\text{sides}(P) - 2).\end{aligned}$$

Alternatively, we can discuss the dual decomposition and consider the curvature at the vertex v'_P. Then since edges are in one-to-one correspondence between the surface and its dual, and as vertices and faces are interchanged, then $\text{sides}(f) = d(v')$, and $d(v) = \text{sides}(f')$. Thus we can re-write our curvature:

$$\begin{aligned}\kappa(P) &= \sum_{v_i \prec P} \frac{2\pi}{d(v_i)} - \pi(\text{sides}(P) - 2) \\ &= \sum_{f'_i \succ v'_P} \frac{2\pi}{\text{sides}(f'_i)} - \pi(d(v'_P) - 2).\end{aligned}$$

This correspondence is illustrated by Figure 4.3, where the original decomposition and angles are in black, while the dual and its angles are in blue. This conception more easily generalises to CW-complexes of dimension higher than two, as then we can assign curvature to the n-cells.

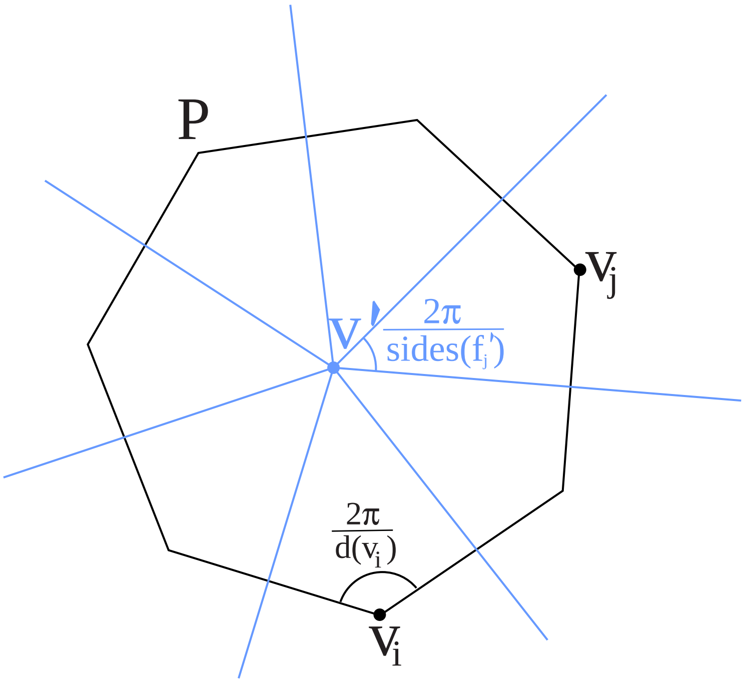

Figure 4.3: Two methods of calculating the geometric curvature of a surface.

We also wish to consider one final conception of geometric curvature, which is equivalent for surfaces, but not otherwise. Instead of allocating the ‘expected’ angles at vertices (with or without an approximation) and then examining whether faces have an excess of angle, we can allocate angles to the faces and look for a deficit at the vertices. That is, for any polygon P, we require that

$$\sum_{v \in P} (\text{angle at } v \text{ in } P) = \pi(\text{sides}(P) - 2),$$

which can be approximated by assuming all angles are equal, and so

$$\text{angle at } v \text{ in } P = \pi - \frac{2\pi}{\text{sides}(P)}.$$

Now we can define the *link geometric curvature* as

$$\hat{\kappa}(P) = 2\pi - \sum_{f \succ v} (\text{angle at } v \text{ in } f).$$

4.2.2 Generalised Geometric Curvature

We now seek to generalise the definitions from Section 4.2.1. Using the dual interpretation, then we can define the *geometric curvature* of a cell $\sigma^{(n)} \in \mathbb{M}$.

Definition 4.2 (Geometric Curvature) *Let $\mathbb{M}$ be an n-dimensional CW-complex. Then for any $\sigma_\alpha^{(n)} \in \mathbb{M}$, with corresponding dual vertex $v'_\alpha \in \mathbb{M}'$, we define the geometric curvature of $\sigma_\alpha^{(n)}$ as:*

$$\begin{aligned} \kappa(\sigma_\alpha^{(n)}) &= \kappa(v'_\alpha) \\ &= \sum_{\substack{2-\text{cells} \\ f' \succ v'}} \frac{2\pi}{\text{sides}(f')} - \pi(d(v') - 2). \end{aligned}$$

If we compare this with the smooth Gaussian curvature (see Definition 1.11), then clearly the two definitions are similarly defined. It is possible, as we did with Forman's Ricci curvature in Section 1.2, to define analogues to sectional curvature (and Ricci curvature) by generalising this definition.

On the other hand, if we used the *link geometric curvature*, then this reduces the discussion of curvature to the link of a vertex (the graph which is dual to the 2-skeleton at v). For any n-manifold $\mathbb{M}$, then this is embedded in the unit (n-1)-sphere, S^{n-1}. This means we can approximate the dihedral angles, namely those between faces at a vertex, by placing the vertices of our link graph on S^{n-1} in some way so that they are as distant as possible. We can approximate by taking some 'good' arrangement (that is one in which the vertices are dispersed) of the vertices on the sphere, and then set all angles to be equal to the minimum distance between any two vertices. This reduces the problem to one where we seek the "max-min", or at least a good lower bound B:

$$\exists ?\, B > 0 : \quad B \leq \max_{\text{configurations}} \left(\min_{\text{pairs } (v_i, v_j)} \text{distance}(v_i, v_j) \right).$$

This is precisely *Tammes' Problem* (see, for example [3]) which has been widely studied in chemistry, physics and biology, as well as mathematics. It is known that the problem can be solved exactly in very few cases (mostly the platonic solids in S^2), but it is unknown (to the author) whether or not there are reasonable lower bounds for the max-min.

When there is additional structure, it would be sensible to use a logical arrangement of the vertices (instead of the random one posited above), as this is likely to yield a better bound (or even the max-min itself). This is possible for the standard lens space triangulations where $L(p, q)$ is formed by identification of two pyramids with base a p-sided polygon. Also, it was shown by Weber and Seifert in [20] that the 3-sphere can be tiled with 120 regular spherical dodecahedra. This decomposition results in the link of each vertex being an icosahedron, and hence in max-min position. Other highly regular examples can be treated in a similar manner.

The primary strength of the link geometric approach is that it can be reconfigured to known problems, and hence may benefit from a large body of prior research. Additionally, it seems likely that subdivision will change the magnitude, but not the sign, of the curvature (as one would expect), and may admit a proof of Myers' Theorem. It should be noted that for low dimensions (two and three, possibly four) the random allocation approximation is particularly bad: we need to 'order' our vertices so that the edges of the link graph are between neighbouring vertices. In higher dimensions there is more room to move, and the approximation ought to be closer to the actual result.

4.3 Some Results on Geometric Curvature

Recall the smooth Gauss-Bonnet Theorem:

Theorem 2.4 (Global Gauss-Bonnet) *Let* Σ *be a closed surface. Then:*

$$\int_\Sigma \boldsymbol{\kappa}\,\mathrm{d}A = 2\pi\chi(\Sigma).$$

We now seek to show that under our definition of geometric curvature, the Gauss-Bonnet theorem holds. First, a preliminary result from graph theory:

Lemma 4.1 (Hand-shaking Lemma) *Let* Γ *be a graph with vertex set* V *and edge set* E*. Then*

$$\sum_{v\in V} d(v) = 2\,|E|.$$

Proof: Take some edge $e \in E$. Then e is incident to precisely two vertices, and hence contributes to two terms (or one term twice) in the sum. Thus

$$\begin{aligned}\sum_{v\in V} d(v) &= 2\sum_{e\in E} 1\\ &= 2\,|E|,\end{aligned}$$

which proves the lemma. □

We are now equipped to prove the Gauss-Bonnet theorem for geometric curvature, an original result:

Theorem 4.2 (Gauss-Bonnet) *Let $\mathbb{M}$ be a 2-dimensional CW-complex with vertices v_α, edges e_β and faces f_γ. Then*

$$\sum_{f_\gamma \in \mathbb{M}} \kappa(f_\gamma) = 2\pi\chi(\mathbb{M}).$$

Proof: Consider the dual manifold $\mathbb{M}'$. Then by definition of the dual (see Definition 1.19), we have that

$$\begin{aligned} v_\alpha &\leftrightarrow f'_\alpha \\ e_\beta &\leftrightarrow e'_\beta \\ f_\gamma &\leftrightarrow v'_\gamma, \end{aligned}$$

and so

$$\begin{aligned} \chi(\mathbb{M}') &= \sum_{p=0}^{2}(-1)^p\#\{p\text{-cells in } \mathbb{M}'\} \\ &= \#\{v'_\gamma \in \mathbb{M}'\} - \#\{e'_\beta \in \mathbb{M}'\} + \#\{f'_\alpha \in \mathbb{M}'\} \\ &= \#\{f_\gamma \in \mathbb{M}\} - \#\{e_\beta \in \mathbb{M}\} + \#\{v_\alpha \in \mathbb{M}\} \\ &= \sum_{p=0}^{2}(-1)^p\#\{p\text{-cells in } \mathbb{M}\} \\ &= \chi(\mathbb{M}). \end{aligned}$$

Now we examine the curvature of some arbitrary face, f_γ. By definition,

$$\begin{aligned} \kappa(f_\gamma) &= \kappa(v'_\gamma) \\ &= \sum_{f' \succ v'_\gamma} \frac{2\pi}{\text{sides}(f')} + \pi(2 - d(v'_\gamma)). \end{aligned}$$

If we now sum over all the dual vertices:

$$\sum_\gamma \kappa(v'_\gamma) = \sum_\gamma \sum_{f' \succ v'_\gamma} \frac{2\pi}{\text{sides}(f')} + \sum_\gamma \pi(2 - d(v'_\gamma)). \tag{4.6}$$

We will deal with each of the two terms separately. Examining the first term in equation 4.6, we can count by faces instead of vertices, and so

$$\begin{aligned} \sum_\gamma \sum_{f' \succ v'_\gamma} \frac{2\pi}{\text{sides}(f')} &= \sum_\alpha \sum_{f'_\alpha \succ v'} \frac{2\pi}{\text{sides}(f')} \\ &= \sum_\alpha \frac{2\pi}{\text{sides}(f')} \cdot \#\{v' \prec f'_\alpha\} \\ &= \sum_\alpha 2\pi \\ &= 2\pi.\#\{f'_\alpha \in \mathbb{M}'\}. \end{aligned}$$

The second term in equation 4.6 gives

$$\sum_{\gamma} \pi(2 - d(v'_\gamma)) = 2\pi.\#\{v'_\gamma \in \mathbb{M}'\} - 2\pi.\#\{e'_\beta \in \mathbb{M}'\},$$

by the Hand-shaking lemma. If we substitute these back into equation 4.6, then

$$\begin{aligned}
\sum_{\gamma} \kappa(f_\gamma) &= \sum_{\gamma} \left[\sum_{f' \succ v'_\gamma} \frac{2\pi}{\text{sides}(f')} + \pi(2 - d(v'_\gamma)) \right] \\
&= 2\pi.\#\{f'_\alpha \in \mathbb{M}'\} + 2\pi.\#\{v'_\gamma \in \mathbb{M}'\} - 2\pi.\#\{e'_\beta \in \mathbb{M}'\} \\
&= 2\pi.\sum_{p=0}^{2} (-1)^p \#\{p\text{-cells in } \mathbb{M}'\} \\
&= 2\pi.\chi(\mathbb{M}') \\
&= 2\pi.\chi(\mathbb{M}),
\end{aligned}$$

which proves the theorem. □

4.4 Future Research

Forman's theory, due to its limitations, is unlikely to yield many practical results in differential geometry. This undermines the purpose to seeking a combinatorial analogue, and hence future research should progress in another direction.

We have proposed an alternate theory of geometric curvature, which is grounded in classical understanding of surfaces. Several methods, which are equivalent in the two-dimensional case, have been discussed; and for which the Gauss-Bonnet Theorem holds (see Theorem 4.2). The primary aspects which need to be assessed as to whether this formulation is actually better than that proposed by Forman are:

(i) whether or not classical examples of positively (negatively) curved spaces are positively (negatively) curved under geometric curvature;

(ii) if subdivision maintains the sign of the geometric curvature; and

(iii) whether or not an analogue of Myers' Theorem can be proved.

Initial examinations indicate that the *link geometric curvature* is the most likely candidate to satisfy all three of these, however there is some subtlety in dealing with low-dimensional cases. This is particularly unfortunate, as these are often the simplest examples to consider. Evidently it will be some time before a widely applicable and useful theory of combinatorial curvature is developed.

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