1 KZ connections

1.1 Connections

Let $\mathcal{O}_X$ be a commutative $\mathbb{C}\text{-}algebra$ with $1$. A derivation of $\mathcal{O}_X$ is a $\mathbb{C}\text{-}linear$ map

$$\partial: \mathcal{O}_X \rightarrow \mathcal{O}_X \quad \text{such that} \quad \partial(fg) = \partial(f)g + f\partial(g),$$

for $f, g \in \mathcal{O}_X$. Let

$$\mathcal{I} = \ker \left( \mathcal{O}_X \otimes \mathcal{O}_X \rightarrow \mathcal{O}_X \right) \quad \text{and} \quad \Omega^1_X = \mathcal{I}/\mathcal{I}^2.$$

Then $\mathcal{O}_X$ acts on $\Omega^1_X$ by

$$f \left( \sum g_i \otimes h_i \right) = \sum fg_i \otimes h_i = \sum g_i \otimes fh_i \mod \mathcal{I}^2,$$

for $f \in \mathcal{O}_X$ and $\sum g_i \otimes h_i \in \mathcal{I}$. Let $\text{Der}(\mathcal{O}_X)$ be the vector space of derivations of $\mathcal{O}_X$. Then

$$d: \mathcal{O}_X \rightarrow \Omega^1_X \otimes \mathcal{O}_X \quad \text{and} \quad \text{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) \xrightarrow{\sim} \text{Der}(\mathcal{O}_X)$$

are $\mathcal{O}_X$ module homomorphisms. If $\Omega^1_X$ is a reflexive $\mathcal{O}_X$ module then $\Omega^1_X = \text{Hom}_{\mathcal{O}_X}(\text{Der}(\mathcal{O}_X), \mathcal{O}_X)$.

Let $M$ be a $\mathcal{O}_X$ module. A connection on $M$ is a $\mathbb{C}\text{-}linear$ map

$$\nabla: M \rightarrow \Omega^1_X \otimes \mathcal{O}_X M \quad \text{such that} \quad \nabla(fm) = d(f) \otimes m + f\nabla(m),$$

for $f \in \mathcal{O}_X$ and $m \in M$. Let $\nabla$ be a connection on $M$ and define

$$\text{Der}(\mathcal{O}_X) \xrightarrow{\partial} \text{End}_{\mathbb{C}}(M) \xrightarrow{\nabla_\partial} \text{by} \quad \nabla_\partial = (\partial \otimes \text{id}_M) \circ \nabla,$$

so that

$$\nabla_\partial: M \xrightarrow{\nabla} \Omega^1_X \otimes \mathcal{O}_X M \xrightarrow{\partial \otimes \text{id}_M} \mathcal{O}_X \otimes \mathcal{O}_X M = M.$$

Then, for $f, g \in \mathcal{O}_X$, $\partial, \partial_1, \partial_2 \in \text{Der}(\mathcal{O}_X)$ and $m \in M$,

(a) $\nabla_\partial(fm) = \partial(f)m + f\nabla_\partial(m),$

(b) $\nabla_{f\partial_1 + g\partial_2}(m) = f\nabla_{\partial_1}(m) + g\nabla_{\partial_2}(m).$

If $\Omega^1_X$ is a reflexive $\mathcal{O}_X$ module then the connection $\nabla$ is determined by the map $\partial \mapsto \nabla_\partial$ with the properties (a) and (b).
1.2 Configuration space

The symmetric group $S_k$ acts on $\mathbb{C}^k$ by permuting coordinates. Then $S_k$ is a reflection group and

$$H_{\varepsilon_i - \varepsilon_j} = \{(z_1, \ldots, z_k) \in \mathbb{C}^k \mid z_i - z_j = 0\}, \quad 1 \leq i < j \leq k,$$

are the reflecting hyperplanes for the reflections in $S_k$. The configuration spaces are

$$\mathcal{D}_k(\mathbb{C}) = \mathbb{C}^k \setminus \bigcup_{1 \leq i < j \leq k} H_{\varepsilon_i - \varepsilon_j} \quad \text{and} \quad C_k(\mathbb{C}) = \mathcal{D}_k(\mathbb{C})/S_k.$$

The braid group is

$$\mathcal{B}_k = \pi_1(\mathcal{C}_k(\mathbb{C})) = \{\text{braids on } k \text{ strands}\}.$$

The pure braid group is

$$\mathcal{P}_B_k = \pi_1(\mathcal{D}_m(\mathbb{C})) = \{\text{braids with } i-th \text{ top dot connected to the bottom } i-th \text{ dot}\},$$

with an exact sequence

$$\{1\} \longrightarrow \mathcal{P}_B_k \longrightarrow \mathcal{B}_k \longrightarrow S_k \longrightarrow \{1\}.$$

1.3 Knizhnik-Zamolodchikov and the classical Yang-Baxter equation

Let $\mathfrak{g}$ be a Lie algebra and let $V_1, \ldots, V_k$ be $\mathfrak{g}$ modules. Let

$$r: \mathbb{C}\setminus\{0\} \longrightarrow \mathfrak{g} \otimes \mathfrak{g} \quad \text{and} \quad z \longmapsto \sum r^{(1)} \otimes r^{(2)}$$

and let

$$r_{ij}(z)(v_1 \otimes \cdots v_k) = v_1 \otimes \cdots v_{j-1} \otimes r^{(1)}v_j \otimes v_{j+1} \otimes \cdots \otimes v_{k-1} \otimes r^{(2)}v_k \otimes v_{k+1} \otimes \cdots \otimes v_m.$$

The KZ-connection (Knizhnik-Zamolodchikov connection) is

$$\nabla = \sum_{1 \leq i < j \leq k} r_{ij}(z_i - z_j)(dz_i - dz_j),$$

a 1-form on $\mathcal{D}_k(\mathbb{C})$ with values in $\text{End}(V_1 \otimes \cdots \otimes V_k)$. Then $\nabla$ defines a connection on the trivial bundle over $\mathcal{D}_k(\mathbb{C})$ with fiber $V_1 \otimes \cdots \otimes V_k$. The connection $\nabla$ is flat if and only if $r$ satisfies the classical Yang-Baxter equation:

$$[r_{12}(z_1 - z_2), r_{23}(z_2 - z_3)] + r_{12}(z_1 - z_2), r_{13}(z_1 - z_3)] + [r_{13}(z_1 - z_3), r_{23}(z_2 - z_3)] = 0.$$

The KZ-equations are

$$\frac{\partial f}{\partial z_j} = \sum_{k=1}^{m} r_{jk}(z_j - z_k)f, \quad \text{for } f: \mathcal{D}_k(\mathbb{C}) \longrightarrow V_1 \otimes \cdots \otimes V_k,$$

the conditions for $f$ to be a covariant flat section of the bundle $\mathcal{D}_k(\mathbb{C}) \times (V_1 \otimes \cdots \otimes V_m)$ with connection $\nabla$. The monodromy of this connection is a representation of the pure braid group $\mathcal{P}_B_k$.

If $V_1 = V_2 = \cdots = V_k$ then $S_k$ acts on $\mathcal{D}_k(\mathbb{C}) \times (V_1 \otimes \cdots \otimes V_k)$ by

$$w((z_1, \ldots, z_k), v_1 \otimes \cdots v_k) = (z_{w(1)}, \ldots z_{w(k)}), v_{w(1)} \otimes \cdots \otimes v_{w(k)}).$$
and \( \nabla \) is \( S_k \) invariant. Thus \( \nabla \) defines a connection in a bundle over \( \mathcal{C}_k(\mathbb{C}) \) with fiber \( V^{\otimes k} \). The monodromy of this connection is a representation of the braid group \( B_k \).

If \( \langle \cdot, \cdot \rangle : g \otimes g \to \mathbb{C} \) is a nondegenerate form and

\[
t = \sum_i b_i \otimes b_i^* ,
\]

where \( \{b_i\} \) is a basis of \( g \) and \( \{b_i^*\} \) is the dual basis of with respect to \( \langle \cdot, \cdot \rangle \) then

\[
r(z_1 - z_2) = \frac{1}{z_1 - z_2} t
\]

satisfies the CYBE (classical Yang-Baxter equation).

1.4 Quantization and KZ

Let \( g \) be a Lie algebra over \( \mathbb{C} \). Let \( g_h \) be a Lie algebra over \( \mathbb{C}[[h]] \) such that \( g_h \cong g[[h]] \) as \( \mathbb{C}[[h]] \) modules. Let

\[
t_h \in S^2(g_h)^G.
\]

There exists a quasitriangular quasiHopf algebra \( A_{g_h, t_h} \) over \( \mathbb{C}[[h]] \) such that \( A_{g_h, t_h} \cong \hat{U}(g_h) \) as \( \mathbb{C}[[h]] \) algebra.

**Proof.** Let \( A_h = \hat{U}(g_h) \) with \( \mathcal{R}_{KZ} = e^{ht_h/2} \).

Note that \( \Delta_{op} \mathcal{R}_{KZ} = \mathcal{R}_{KZ} \Delta_h \mathcal{R}_{KZ}^{-1} \) by the \( g_h \) invariance of \( t_h \). The *universal KZ equation* is

\[
\frac{\partial f}{\partial z_j} = \hbar \sum_{k=1}^{m} \frac{j_{jk}^h}{z_j - z_k} f, \quad j = 1, 2, \ldots, m, \quad (KZ_m)
\]

where \( h = \frac{\hbar}{2\pi} \) and \( f : \mathbb{C}^m \to (U(g_h))^{\otimes m} \). Note that

\[
\mathcal{R}_{KZ} = e^{ht_h/2}
\]

is the monodromy of \((KZ_2)\).

Define

\[
\Phi_{KZ} = g_1^{-1} g_2 \quad \text{in} \quad (U(g_h))^{\otimes 3}
\]

by analytic solutions \( g_1 : \mathbb{C} \to (U(g_h))^{\otimes 3} \) and \( g_2 : \mathbb{C} \to (U(g_h))^{\otimes 3} \) of the equation

\[
g'(x) = \hbar \left( \frac{t_{12}^h}{x} + \frac{t_{23}^h}{x - 1} \right) g(x),
\]

so that

\[
f(z_1, z_2; z_3) = (z_3 - z_1) h(t_{12}^h + t_{13}^h + t_{23}^h) g \left( \frac{z_2 - z_1}{z_3 - z_1} \right)
\]

is a solution of \((KZ_3)\) (using the invariance of \((KZ_3)\) under transformations \( z_i \mapsto az_i + b \)). Here \( g_1 \) and \( g_2 \) have asymptotic behaviour

\[
g_1(x) \sim x^{ht_{12}^h} \quad \text{as} \quad x \to 0
\]
\[
g_2(x) \sim (1 - x)^{ht_{23}^h} \quad \text{as} \quad x \to 1.
\]
Then \((\hat{U}g[[h]], \Delta, \mathcal{R}_{KZ}, \Phi_{KZ})\) is a quasi Hopf algebra.

We want a Hopf algebra, i.e. a quasi Hopf algebra with \(\Phi_h = 1 \otimes 1 \otimes 1\). The point is that one can twist to
\[
(U_h g, \mathcal{R}_h, \Phi_h = \text{id}), \quad \text{where } U_h g \text{ is the quantum group},
\]
i.e. there exists \(F \in Ug[[h]] \otimes Ug[[h]]\) such that
\[
\begin{align*}
\Delta_h &= F \Delta(h) F^{-1}, \\
\Phi_h &= \mathcal{F}_{12}(\Delta \otimes \text{id})(\mathcal{F}) \Phi_{KZ}(\text{id} \otimes \Delta)(\mathcal{F})^{-1} \mathcal{F}_{23}^{-1}, \\
\mathcal{R}_h &= \mathcal{F}_{21} \mathcal{R}_{KZ} \mathcal{F}_{12}^{-1}.
\end{align*}
\]

\[\square\]

### 1.5 Affine Lie algebras and KZ

Let \(g = n^- \oplus \mathfrak{h} \oplus n^+\) be a finite dimensional simple Lie algebra over \(\mathbb{C}\) and fix a nondegenerate symmetric \(\text{ad}\)-invariant bilinear form \((\cdot, \cdot)\) on \(g\). The **affine Lie algebra** is the Lie algebra
\[
\hat{g} = g((z)) \oplus \mathbb{C} c
\]
with \(c\) central and with
\[
[f, g] = [f(z), g(z)] + \text{res}_0 \left( \langle f(z), \frac{dg}{dz} \rangle \right) \cdot c,
\]
where \(\text{res}_0 h\) is the coefficient of \(z^{-1}\) in \(h\). Let \(d\) be the derivation of \(\hat{g}\) such that
\[
[d, c] = 0 \quad \text{and} \quad [d, f] = z \frac{df}{dz}.
\]

Let \(V\) be a finite dimensional irreducible \(g\) module. The **loop representation** is the \(\hat{g}\) module
\[
V((z)) = V \otimes \mathbb{C}((z)), \quad \text{with } c \text{ acting by } 0.
\]

A **level} \(k\) highest weight representation of highest weight \(\lambda\) is a \(\hat{g}\) module \(W\) with a vector \(w^+ \in W\) such that
\[
\begin{align*}
(a) \quad W &= (U\hat{g}) w^+, \\
(b) \quad cw^+ &= \kappa w^+, \\
(c) \quad zg[[z]]w^+ &= 0 \text{ and } n^+ w^+ = 0, \\
(d) \quad hw^+ &= \lambda(h) w^+, \text{ for } h \in \mathfrak{h}.
\end{align*}
\]

The **conformal weight** of \(W\) is
\[
h_{\kappa}(\lambda) = \frac{\langle \lambda + \rho, \lambda + \rho \rangle - \langle \rho, \rho \rangle}{2(\kappa + h^V)}
\]

Let \(W_0, W_1\) be highest weight representations of level \(\kappa\) and highest weights \(\lambda^{(0)}\) and \(\lambda^{(1)}\). Let \(V\) be a simple \(g\) module of highest weight \(\mu\). An **intertwining operator** is a \(\hat{f}g\) module homomorphism
\[
I_{W_1, W_0}^V : W_1 \to W_0 \otimes V((z)) \quad \text{and} \quad J_{V}^{W_1, W_0} = z^{h_{\kappa}(\lambda^{(0)}) + h_{\kappa}(\mu) - h_{\kappa}(\lambda^{(1)})} I_{V}^{W_1, W_0}
\]
is the rescaled intertwining operator. Let $W_0, \ldots, W_k$ be highest weight $\hat{\mathfrak{g}}$ modules of level $\kappa$ and let $V_1, \ldots, V_k$ be simple $\mathfrak{g}$ modules. Let $F = T_{W_0, W_1} \circ \cdots \circ T_{W_{k-1}, W_k}$ so that

$$F: W_k \to W_{k-1} \otimes V_k((z_k)) \to W_{k-2} \otimes V_{k-1}((z_{k-1})) \otimes V_k((z_k)) \to \cdots \to W_0 \otimes V_1((z_1)) \otimes \cdots \otimes V_k((z_k)).$$

Then define

$$f: D_k(\mathbb{C}) \to V_1 \otimes \cdots \otimes V_k$$

by

$$f(z_1, \ldots, z_m) = \langle w_0^+, Fw_k^+ \rangle.$$

Then $f$ satisfies the KZ equations for

$$r(z_1 - z_2) = \frac{1}{\kappa + h^\vee} \frac{t}{z_1 - z_2}.$$

### 1.6 Affine Lie algebras and quantum groups

Let

$$\tilde{\mathfrak{g}} = \mathfrak{g}[z, z^{-1}],$$

a subalgebra of $\hat{\mathfrak{g}}$.

If $V$ is a $\tilde{\mathfrak{g}}$ module then

$$V(1) \subseteq V(2) \subseteq \cdots$$

where

$$V(N) = \{ v \in V \mid (z\mathfrak{g})^N v = 0 \}.$$

The smooth vectors in $V$ are the elements of

$$V(\infty) = \bigcup_{N \in \mathbb{Z}_{>0}} V(N).$$

The category $\mathcal{O}_\kappa$ is the category of $\tilde{\mathfrak{g}}$ modules $V$ such that

(a) $c$ acts by the scalar $\kappa - h$, where $h$ is the Coxeter number,

(b) If $v \in V$ then $\dim((U\mathfrak{g})v)$ is finite,

(c) If $v \in V$ and $x \in z\mathfrak{g}[[z]]$ then $x^N v = 0$ for $N >> 0$,

(d) $V$ is a finitely generated $\tilde{\mathfrak{g}}$ module.

Equivalently $\mathcal{O}_\kappa$ is the category of smooth $\tilde{\mathfrak{g}}$ modules such that

$$c$$

acts by $\kappa - h$ and $\dim(V(1))$ is finite.

Let $q = e^{-i\pi \kappa}$ and let $\tilde{\mathcal{O}}_\kappa$ be the category of finite dimensional $U_q \tilde{\mathfrak{g}}$ modules of type 1 which are $U_0$ semisimple.

**Theorem 1.1.** There is an equivalence of categories

$$\mathcal{O}_\kappa \cong \tilde{\mathcal{O}}_\kappa.$$

Let $V(\lambda)$ be the irreducible finite dimensional $\mathfrak{g}$ module of highest weight $\lambda \in \mathcal{P}^+$. Extend $V(\lambda)$ to a $\tilde{\mathfrak{g}}[[z]]$ module by letting $z\mathfrak{g}[[z]]$ act trivially and let $c$ act by $(\kappa - h)$. The Weyl module is the $\tilde{\mathfrak{g}}$ module given by

$$W^\kappa(\lambda) = U\tilde{\mathfrak{g}} \otimes_{U(\tilde{\mathfrak{g}}[[z]]) \otimes \mathbb{C}^0} V(\lambda).$$

Let

$$L^\kappa(\lambda)$$

be the unique simple quotient of $W^\kappa(\lambda)$. 5