The affine Weyl group

Arun Ram
Department of Mathematics
University of Wisconsin
Madison, WI 53706
ram@math.wisc.edu

1 The affine Weyl group

This section is a summary of the main facts and notations that are needed for working with the affine Weyl group $\tilde{W}$. The main point is that the elements of the affine Weyl group can be identified with alcoves via the bijection in (2.11).

Let $h^*_R$ be a finite dimensional vector space over $\mathbb{R}$. A reflection is a diagonalizable element of $GL(h^*_R)$ which has exactly one eigenvalue not equal to 1. A lattice is a free $\mathbb{Z}$-module. A Weyl group is a finite subgroup $W$ of $GL(h^*_R)$ which is generated by reflections and acts on a lattice $L$ in $h^*_R$.

The chambers are the connected components of the complement $h^*_R \setminus (\bigcup_{\alpha \in R^+} H_{\alpha})$ of these hyperplanes in $h^*_R$. These are fundamental regions for the action of $W$.

Let $\langle , \rangle$ be a nondegenerate $W$-invariant bilinear form on $h^*_R$. Fix a chamber $C$ and choose vectors $\alpha^\vee \in h^*_R$ such that

$$C = \{x \in h^*_R \mid \langle x, \alpha^\vee \rangle > 0\} \quad \text{and} \quad P \supseteq L \supseteq Q,$$

(1.1)

where

$$P = \{\lambda \in h^*_R \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\} \quad \text{and} \quad Q = \sum_{\alpha \in R^+} \mathbb{Z}\alpha, \quad \text{where} \quad \alpha = \frac{2\alpha^\vee}{\langle \alpha^\vee, \alpha^\vee \rangle}.$$

(1.2)

Pictorially, $\langle \lambda, \alpha^\vee \rangle$ is the distance from $\lambda$ to the hyperplane $H_\alpha$.

The alcoves are the connected components of the complement

$$h^*_R \setminus (\bigcup_{\alpha \in R^+, j \in \mathbb{Z}} H_{\alpha,j})$$

of the (affine) hyperplanes $H_{\alpha,j} = \{x \in h^*_R \mid \langle x, \alpha^\vee \rangle = j\}$.
in \( h^*_R \). The fundamental alcove is the alcove
\[
A \subseteq C \quad \text{such that} \quad 0 \in \overline{A},
\]
where \( \overline{A} \) is the closure of \( A \). An example is the case of type \( C_2 \), where the picture is

\[
\begin{align*}
H_{\varphi} &= H_{\alpha_1 + \alpha_2} \\
H_{\alpha_1 + \alpha_2, -1} &\quad H_{\alpha_1 + \alpha_2, 1} \\
H_{\alpha_1 + \alpha_2, 0} &\quad H_{\alpha_2, 0} = H_{\alpha_2} \\
&\quad H_{\alpha_2, 2} \\
&\quad H_{\alpha_2, -2} \\
&\quad \cdots \\
&\quad H_{\alpha_2, 1} \\
&\quad \cdots \\
&\quad H_{\alpha_2, -5} \\
&\quad \cdots \\
&\quad H_{\alpha_2, 3} \\
&\quad \cdots \\
&\quad H_{\alpha_2, 4} \\
&\quad \cdots \\
&\quad H_{\alpha_2, 5} \\
&\quad \cdots
\end{align*}
\]

The translation in \( \lambda \) is the operator \( t_\lambda : h^*_R \rightarrow h^*_R \) given by
\[
t_\lambda(x) = \lambda + x, \quad \text{for} \quad \lambda \in P, x \in h^*_R.
\]

The reflection \( s_{\alpha, k} \) in the hyperplane \( H_{\alpha, k} \) is given by
\[
s_{\alpha, k} = t_{k\alpha}s_{\alpha} = s_{\alpha}t_{-k\alpha}.
\]

The extended affine Weyl group is
\[
\widetilde{W} = P \rtimes W = \{ t_\lambda w \mid \lambda \in P, w \in W \} \quad \text{with} \quad wt_\lambda = t_{w\lambda}w.
\]

Denote the walls of \( C \) by \( H_{\alpha_1}, \ldots, H_{\alpha_n} \) and extend this indexing so that
\[
H_{\alpha_0}, \ldots, H_{\alpha_n} \quad \text{are the walls of} \quad A,
\]
the fundamental alcove. Then the affine Weyl group,
\[
W_{\text{aff}} = Q \rtimes W \quad \text{is generated by} \quad s_0, \ldots, s_n,
\]
the reflections in the hyperplanes \( H_{\alpha_0}, \ldots, H_{\alpha_n} \). Furthermore, \( A \) is a fundamental region for the action of \( W_{\text{aff}} \) on \( h^*_R \) and so there is a bijection
\[
\begin{align*}
W_{\text{aff}} &\longrightarrow \{ \text{alcoves in } h^*_R \} \\
w &\mapsto w^{-1}A.
\end{align*}
\]

The length of \( w \in \widetilde{W} \) is
\[
\ell(w) = \text{number of hyperplanes between } A \text{ and } wA.
\]
The difference between $W_{\text{aff}}$ and $\widetilde{W}$ is the group

$$\Omega = \widetilde{W}/W_{\text{aff}} \cong P/Q. \quad (1.9)$$

The group $\Omega$ is the set of elements of $\widetilde{W}$ of length $0$. An element of $\Omega$ acts on the fundamental alcove $A$ by an automorphism. Its action on $A$ induces a permutation of the walls of $A$, and hence a permutation of $0, 1, \ldots, n$. If $g \in \Omega$ and $g \neq 1$ let $\omega_i$ be the image of the origin under the action of $g$ on $A$. If $s_j$ denotes the reflection in the $j$th wall of $A$ and $w_i$ denotes the longest element of the stabilizer $W_{\omega_i}$ of $\omega_i$ in $W$, then

$$gs_ig^{-1} = s_{g(i)} \quad \text{and} \quad gw_0w_i = t_{\omega_i}. \quad (1.10)$$

The group $\widetilde{W}$ acts freely on $\Omega \times h_\mathbb{R}^n (|\Omega| \text{ copies of } \mathbb{R}^n \text{ tiled by alcoves})$ so that $g^{-1}A$ is in the same spot as $A$ except on the $g$th “sheet” of $\Omega \times h_\mathbb{R}^n$. It is helpful to think of the elements of $\Omega$ as the deck transformations which transfer between the sheets in $\Omega \times h_\mathbb{R}^n$. Then

$$\widetilde{W} \quad \xrightarrow{w} \quad \{\text{ alcoves in } \Omega \times h_\mathbb{R}^n\} \quad w^{-1}A \quad (1.11)$$

is a bijection. In type $C_2$, the two sheets in $\Omega \times h_\mathbb{R}^n$ look like
where the numbering on the walls of the alcoves is \( \widetilde{W} \) equivariant so that, for \( w \in \widetilde{W} \), the numbering on the walls of \( wA \) is the \( w \) image of the numbering on the walls of \( A \).

The \textit{0-polygon} is the \( W \)-orbit of \( A \) in \( \Omega \times \mathfrak{h}^*_R \) and for \( \lambda \in P \), the \( \lambda \)-\textit{polygon} is \( \lambda + WA \),

the translate of the \( W \) orbit of \( A \) by \( \lambda \). The space \( \Omega \times \mathfrak{h}^*_R \) is tiled by the polygons and, via (2.11), we make identifications between \( W, \widetilde{W} \), \( P \) and their geometric counterparts in \( \Omega \times \mathfrak{h}^*_R \):

\[
\widetilde{W} = \{\text{alcoves}\}, \quad W = \{\text{alcoves in the 0-polygon}\}, \quad P = \{\text{centers of polygons}\}. \quad (1.13)
\]

Define

\[
P^+ = P \cap C \quad \text{and} \quad P^{++} = P \cap C \quad (1.14)
\]

so that \( P^+ \) is a set of representatives of the orbits of the action of \( W \) on \( P \). The \textit{fundamental weights} are the generators \( \omega_1, \ldots, \omega_n \) of the \( \mathbb{Z}_{\geq 0} \)-module \( P^+ \) so that

\[
C = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} \omega_i, \quad P^+ = \sum_{i=1}^{n} \mathbb{Z}_{\geq 0} \omega_i, \quad \text{and} \quad P^{++} = \sum_{i=1}^{n} \mathbb{Z}_{>0} \omega_i. \quad (1.15)
\]

The lattice \( P \) has \( \mathbb{Z} \)-basis \( \omega_1, \ldots, \omega_n \) and the map

\[
P^+ \quad \longrightarrow \quad P^{++} \quad \text{where} \quad \rho = \omega_1 + \ldots + \omega_n, \quad (1.16)
\]
is a bijection. The simple coroots are $\alpha_1^\vee, \ldots, \alpha_n^\vee$ the dual basis to the fundamental weights,

$$\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}. \quad (1.17)$$

Define

$$C^\vee = \sum_{i=1}^n \mathbb{R}_{\leq 0} \alpha_i^\vee \quad \text{and} \quad C^\vee = \sum_{i=1}^n \mathbb{R}_{< 0} \alpha_i^\vee. \quad (1.18)$$

The dominance order is the partial order on $h^*_\mathbb{R}$ given by

$$\mu \leq \lambda \quad \text{if} \quad \mu \in \lambda + C^\vee. \quad (1.19)$$

In type $C_2$ the lattice $P = \mathbb{Z} \varepsilon_1 + \mathbb{Z} \varepsilon_2$ with $\{\varepsilon_1, \varepsilon_2\}$ an orthonormal basis of $h^*_\mathbb{R} \cong \mathbb{R}^2$ and $W = \{s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2\}$ is the dihedral group of order 8 generated by the reflections $s_1$ and $s_2$ in the hyperplanes $H_{\alpha_1}$ and $H_{\alpha_2}$, respectively, where

$$H_{\alpha_1} = \{x \in h^*_\mathbb{R} \mid \langle x, \varepsilon_1 \rangle = 0\} \quad \text{and} \quad H_{\alpha_2} = \{x \in h^*_\mathbb{R} \mid \langle x, \varepsilon_2 - \varepsilon_1 \rangle = 0\}.$$
In this case
\[
\omega_1 = \varepsilon_1 + \varepsilon_2, \quad \alpha_1 = 2\varepsilon_1, \quad \alpha_1^{\vee} = \varepsilon_1,
\]
\[
\omega_2 = \varepsilon_2, \quad \alpha_2 = \varepsilon_2 - \varepsilon_1, \quad \alpha_2^{\vee} = \alpha_2,
\]
and
\[
R = \{\pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2), \pm (\alpha_1 + 2\alpha_2)\}.
\]