1 The affine Hecke algebra

1.1 The alcove walk algebra

Fix notations for the Weyl group $W$, the extended affine Weyl group $\tilde{W}$, and their action on $\Omega \times h^*_\R$ as in Section 2. Label the walls of the alcoves so that the fundamental alcove has walls labeled $0, 1, \ldots, n$ and the labeling is $\tilde{W}$-equivariant (see the picture in (2.12)).

The periodic orientation is the orientation of the walls of the alcoves given by

setting the positive side of $H_{\alpha,j}$ to be $\{x \in h^*_\R \mid \langle x, \alpha^\vee \rangle > j\}$. \hspace{1cm} (1.1)

This is an orientation of the walls of the alcoves such that if $\Delta$ is an alcove and $\lambda \in P$ then

the walls of $\lambda + \Delta$ have the same orientation as the walls of $\Delta$.

Let $\K$ be a field. Use the notations for elements of $\Omega$ as in (2.10). The alcove walk algebra is the algebra over $\K$ given by generators $g \in \Omega$ and

\begin{align*}
\begin{array}{cccc}
- & i & + & - & i & + & - & i & + & - & i & + \\
\text{positive } i\text{-crossing} & \quad \text{negative } i\text{-crossing} & \quad \text{positive } i\text{-fold} & \quad \text{negative } i\text{-fold}
\end{array}
\end{align*}

(1 \leq i \leq n)

with relations (straightening laws)

\begin{align*}
\begin{array}{c}
- & i & + & = & - & i & + & - & i & + & - & i & + \\
\text{and} & \quad & - & i & + & = & - & i & + & - & i & +
\end{array} \hspace{1cm} & \hspace{1cm} (1.2)
\end{align*}

and

\begin{align*}
g \begin{pmatrix}
- & i & + \\
\end{pmatrix} & = \begin{pmatrix}
g(i) & \\
- & i & + \\
\end{pmatrix} g, \\
g \begin{pmatrix}
- & i & + \\
\end{pmatrix} & = \begin{pmatrix}
g(i) & \\
- & i & + \\
\end{pmatrix} g, \\
g \begin{pmatrix}
- & i & + \\
\end{pmatrix} & = \begin{pmatrix}
g(i) & \\
- & i & + \\
\end{pmatrix} g.
\end{align*}
Viewing the product as concatenation each word in the generators can be represented as a sequence of arrows, with the first arrow having its head or its tail in the fundamental alcove. An alcove walk is a word in the generators such that,

(a) the tail of the first step is in the fundamental alcove $A$,
(b) at every step, the head of each arrow is in the same alcove as the tail of the next arrow.

The type of a walk $p$ is the sequence of labels on the arrows. Note that, if $w \in \tilde{W}$ then

$$\ell(w) = \text{length of a minimal length walk from } A \text{ to } wA. \quad (1.3)$$

For example, in type $C_2$,

$$H_{\alpha_1+\alpha_2} \quad H_{\alpha_1} \quad H_{\alpha_2} \quad H_{\alpha_1+2\alpha_2}$$

is an alcove walk $p$ of type $(1, 2, 0, 1, 0, 2, 1, 2, 1, 0, 1, 2)$ with two folds. Using the notation

$$c_i^+ \text{ for a positive } i\text{-crossing,} \quad f_i^+ \text{ for a positive } i\text{-fold,}$$
$$c_i^- \text{ for a negative } i\text{-crossing,} \quad f_i^- \text{ for a negative } i\text{-fold,} \quad (1.4)$$

the walk in the picture is $c_1^+ c_2^- c_i^+ f_i^+ c_1^+ c_2^- f_i^- c_i^+ c_2^+ c_1^- c_2^-$.  

The proof of the following lemma is straightforward following the scheme indicated by the example which follows.

**Lemma 1.1.** The set of alcove walks is a basis of the alcove walk algebra.

For example, in type $C_2$, a product of the generators which is not a walk is

$$c_1^- c_2^+ c_0^+ c_1^- f_0^- c_2^+ c_1^- c_2^+ f_1^- c_0^- c_1^+ c_2^-.$$
but, by first applying relations $f_i^\pm = - f_i^\mp$ and then working left to right applying the relations
\[ c_i^+ = c_i^- + f_i^\pm, \]
gives
\[
c_1 c_2 c_0 c_1 f_0 c_2 c_1 c_2 f_1 c_0 c_1 c_2 = -(c_1 c_2 c_0 c_1 f_0 c_2 c_1 c_2 f_1 c_0 c_1 c_2) \\
= -(c_1 (c_2 + f_2) c_0 c_1 f_0 c_2 c_1 c_2 f_1 c_0 c_1 c_2) \\
= -(c_1 (c_2 + f_2) c_0 c_1 f_0 c_2 (c_1 + f_1) c_2 f_1 c_0 c_1 c_2) \\
= -(c_1 (c_2 + f_2) c_0 c_1 f_0 c_2 (c_1 + f_1) c_2 f_1 c_0 c_1 c_2 (c_2 + f_2))
\]
and every term in the expansion of this expression is an alcove walk.

1.2 The affine Hecke algebra

Fix an invertible element $q \in \mathbb{K}$. The **affine Hecke algebra** $\tilde{H}$ is the quotient of the alcove walk algebra by the relations

\[
\begin{align*}
- i^+ &= \left( \begin{array}{c} - i^+ \\ - i^+ \end{array} \right)^{-1}, \\
- i^- &= -(q - q^{-1}), \\
- i^+ &= (q - q^{-1}),
\end{align*}
\]

and

\[
p = p' \quad \text{if } p \text{ and } p' \text{ are nonfolded walks with } \text{end}(p) = \text{end}(p'),
\]

where end($p$) is the final alcove of $p$. Conceptually, the affine Hecke algebra only remembers the ending alcove of a walk (and some information about the folds) and forgets how it got to its destination.

For $w \in W$ and $\lambda \in P$ define elements

\[
\begin{align*}
T_{w^{-1}} &= \text{(image in } \tilde{H} \text{ of a minimal length alcove walk from } A \text{ to } wA),} \\
X^\lambda &= \text{(image in } \tilde{H} \text{ of a minimal length alcove walk from } A \text{ to } t_\lambda A).
\end{align*}
\]
The following proposition shows that the alcove walk definition of the affine Hecke algebra coincides with the standard definition by generators and relations (see [IM] and [Lu]). A consequence of the proposition is that

\[ H = \text{span}\{T_{w^{-1}}^{-1} \mid w \in W\}, \quad \text{and} \quad \mathbb{K}[P] = \text{span}\{X^\lambda \mid \lambda \in P\}, \tag{1.6} \]

are subalgebras of \( \tilde{H} \).

**Proposition 1.2.** Let \( g \in \Omega, \, \lambda, \mu \in P, \, w \in W \) and \( 1 \leq i \leq n \). Let \( \varphi \) be the element of \( R^+ \) such that \( H_{\alpha_0} = H_{\varphi,1} \) is the wall of \( A \) which is not a wall of \( C \) and let \( s_\varphi \) be the reflection in \( H_{\varphi} \). Let \( w_0 \) be the longest element of \( W \). The following identities hold in \( \tilde{H} \).

(a) \( X^\lambda X^\mu = X^{\lambda+\mu} = X^\mu X^\lambda \).

(b) \( T_{s_i} T_w = \begin{cases} T_{s_i w}, & \text{if } \ell(s_i w) > \ell(w), \\ T_{s_i w} + (q-q^{-1})T_w, & \text{if } \ell(s_i w) < \ell(w). \end{cases} \)

(c) If \( \langle \lambda, \alpha_i^\vee \rangle = 0 \) then \( T_{s_i} X^\lambda = X^\lambda T_{s_i} \).

(d) If \( \langle \lambda, \alpha_i^\vee \rangle = 1 \) then \( T_{s_i} X^{\lambda+\alpha_i} = X^\lambda. \)

(e) \( T_{s_i} X^\lambda = X^{s_i \lambda} T_{s_i} + (q-q^{-1})X^\lambda - X^{s_i \lambda} \frac{1}{1 - X^{-\alpha_i}}. \)

(f) \( T_{s_0} T_{s_\varphi} = X^\varphi. \)

(g) \( X^{\omega_i} = gT_{w_0 w_i} \), where the action of \( g \) on \( A \) sends the origin to \( \omega_i \) and \( w_i \) is the longest element of the stabilizer \( W_{\omega_i} \) of \( \omega_i \) in \( W \).

**Proof.** Use notations for alcove walks as in (3.4).

(a) If \( p_\lambda \) is a minimal length walk from \( A \) to \( t_\lambda A \) and \( p_\mu \) is a minimal length walk from \( A \) to \( t_\mu A \) then

\[ p_\lambda p_\mu \quad \text{and} \quad p_\mu p_\lambda \quad \text{are both nonfolded walks from } A \text{ to } t_{\lambda+\mu} A. \]

Thus the images of \( p_\lambda p_\mu \) and \( p_\mu p_\lambda \) are equal in \( \tilde{H} \).

(b) If \( \ell(ws_i) > \ell(w) \) and \( p_w \) is a minimal length walk from \( A \) to \( wA \) then

\[ p_{ws_i} = p_w c_i^- \quad \text{is a minimal length walk from } A \text{ to } ws_i A. \]

and so \( T_{s_i w}^{-1} = T_{ws_i}^{-1} = T_{w^{-1}}^{-1} T_{s_i}^{-1} = (T_{s_i} T_{w^{-1}})^{-1} \) in \( \tilde{H} \). Taking inverses gives the first result, and the second follows by switching \( w \) and \( ws_i \) and using the relation \( T_{s_i}^{-1} = T_{s_i} - (q-q^{-1}) \) which follows from (3.2) and (3.5).
(c) Let $p_{\lambda}$ be a minimal length alcove walk from $A$ to $t_{\lambda}A$. If $\langle \lambda, \alpha_i^\vee \rangle = 0$ then $H_{\alpha_i}$ is a wall of $t_{\lambda}A$ and $s_i \lambda = \lambda$ and $c_i^- p_{\lambda} c_i^+$ is a nonfolded walk from $A$ to $t_{\lambda}A$.

Thus $T_{s_i}^{-1} X^\lambda T_{s_i} = X^\lambda = X^{s_i \lambda}$ in $\tilde{H}$.

(d) Let $p_{\lambda}$ be a minimal length walk from $A$ to $t_{\lambda}A$. If $\langle \lambda, \alpha_i^\vee \rangle = 1$ then there is a minimal length walk from $A$ to $t_{\lambda}A$ of the form $p_{\lambda} = p_{i_1 \cdots i_r} s_i^{\alpha_i^+}$ where $p_{i_1 \cdots i_r}$ is minimal length walk from $A$ to $t_{\lambda} s_i A$. Then $c_i^- p_{i_1 \cdots i_r}$ is a minimal length walk from $A$ to $t_{s_i \lambda}A$.

Thus $T_{s_i}^{-1} (X^\lambda T_{s_i}^{-1}) = X^{s_i \lambda}$ in $\tilde{H}$. 

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(e) Note that (c) and (d) are special cases of (e). If the statement of (e) holds for \( \lambda \) then, by multiplying on the left by \( X^{-s_i} \lambda \) and on the right by \( X^{-\lambda} \), it holds for \(-\lambda\). If the statement (e) holds for \( \lambda \) and \( \mu \) then it holds for \( \lambda + \mu \) since

\[
T_sX^\lambda X^\mu = \left(X^{s_i\lambda}T_s + (q - q^{-1})\frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}\right)X^\mu
\]

\[
= X^{s_i\lambda}\left(X^{s_i\mu}T_s + (q - q^{-1})\frac{X^\mu - X^{s_i\mu}}{1 - X^{-\alpha_i}}\right) + (q - q^{-1})\left(\frac{X^\lambda - X^{s_i\lambda}}{1 - X^{-\alpha_i}}\right)X^\mu
\]

\[
= X^{s_i(\lambda + \mu)}T_s + (q - q^{-1})\frac{X^{\lambda + \mu} - X^{s_i(\lambda + \mu)}}{1 - X^{-\alpha_i}}.
\]

Thus, to prove (e) it is sufficient to verify (c) and (d), which has already been done.

(f) Let \( p_{s\varphi} \) be a minimal length walk from \( s\varphi A \) to \( A \), then

\[
p_{\varphi} = c_0^+ p_{s\varphi} \text{ is a minimal length walk from } A \text{ to } t_{\varphi}A.
\]

Thus \( T_0T_{s\varphi} = X^\varphi \) in \( \tilde{H} \).

(g) If \( p_{w_0w_i} \) is a minimal length walk from \( w_iw_0A \) to \( A \)

\[
p_{w_i} = gp_{w_0w_i} \text{ is a minimal length walk from } A \text{ to } t_{w_i}A.
\]

Thus \( X^{w_i} = gT_{w_0w_i} \) in \( \tilde{H} \). For example, in type \( C_2 \), \( w_0 = s_2s_1s_2s_1 \) and there is one element \( g \) in \( \Omega \) such that \( g \varnothing_2 = 0 \) and \( w_2 = s_1 \) so that \( w_0w_2 = s_2s_1s_2 \).

The sets

\[
\{T^{-1}_{w^{-1}}X^\lambda \mid w \in W, \lambda \in P\} \quad \text{and} \quad \{X^\mu T^{-1}_{w^{-1}} \mid \mu \in P, v \in W\}
\]

are bases of \( \tilde{H} \). If \( p \) is an alcove walk then the weight of \( p \) and the final direction of \( p \) are

\[
\text{wt}(p) \in P \text{ and } \varphi(p) \in W \quad \text{such that} \quad p \text{ ends in the alcove } \text{wt}(p) + \varphi(p)A.
\]

Let

\[
\begin{align*}
f^-(p) &= \text{(number of negative folds of } p), \\
f^+(p) &= \text{(number of positive folds of } p), \\
f(p) &= \text{(total number of folds of } p).
\end{align*}
\]

The following theorem provides a combinatorial formulation of the transition matrix between the bases in (3.7). It is a \( q \)-version of the main result of [LP] and an extension of Corollary 6.1 of [Sc].

**Theorem 1.3.** Use notations as in (3.4). Let \( \lambda \in P \) and \( w \in W \). Fix a minimal length walk \( p_w = c_{i_1}^+ c_{i_2}^+ \cdots c_{i_r}^+ \) from \( A \) to \( wA \) and a minimal length walk \( p_\lambda = c_{j_1}^- \cdots c_{j_s}^- \) from \( A \) to \( t_{\lambda}A \). Then, with notations as in (3.8) and (3.9),

\[
T^{-1}_{w^{-1}}X^\lambda = \sum_p (-1)^{f^-(p)}(q - q^{-1})f(p)X^{\text{wt}(p)}T^{-1}_{p^{-1}}\varphi(p)^{-1},
\]

where the sum is over all alcove walks \( p = c_{i_1}^- \cdots c_{i_r}^-p_{j_1} \cdots p_{j_s} \) such that \( p_{j_k} \) is either \( c_{j_k}^+ \), \( c_{j_k}^- \) or \( f_{j_k}^\pm \).
Proof. The product \( p_w p_\lambda = c_i^\epsilon_i c_j^\epsilon_j \cdots c_1^\epsilon_1 c_s^\epsilon_s \) may not necessarily be walk, but its straightening produces a sum of walks, and this decomposition gives the formula in the statement. \( \square \)

Remark 1.4. The initial direction \( \iota(p) \) and the final direction \( \varphi(p) \) of an alcove walk \( p \) appear naturally in Theorem 3.3. These statistics also appear in the Pieri-Chevalley formula in the \( K \)-theory of the flag variety (see [PR], [GR], [Br] and [LP]).

Remark 1.5. In Theorem 3.3, for certain \( \lambda \) the walk \( p_\lambda \) may be chosen so that all the terms in the expansion of \( T_{w^{-1}} X^\lambda \) have the same sign. For example, if \( \lambda \) is dominant, then \( p_\lambda \) can be taken with all \( \epsilon_k = + \), in which case all folds which appear in the straightening of \( p_w p_\lambda \) will be positive folds and so all terms in the expansion will be positive. If \( \lambda \) is antidominant then \( p_\lambda \) can be taken with all \( \epsilon_k = - \) and all terms in the expansion will be negative. This fact gives positivity results for products in the cohomology and the \( K \)-theory of the flag variety (see [PR], [Br]).

Remark 1.6. The affine Hecke algebra \( \tilde{H} \) has basis \( \{X^\lambda T_{w^{-1}}^{-1} \mid \lambda \in P, w \in W\} \) in bijection with the alcoves in \( \Omega \times h^*_\mathbb{R} \), where \( X^\lambda T_{w^{-1}}^{-1} \) is the image in \( \tilde{H} \) of a minimal length alcove walk from \( A \) to the alcove \( \lambda + wA \). Changing the orientation of the walls of the alcoves changes the resulting basis in the affine Hecke algebra \( \tilde{H} \). The orientation in (3.1) is the one such that

\[
\text{the most negative point is } -\infty \rho, \text{ deep in the chamber } w_0C. \tag{1.10}
\]

Another standard orientation is where

\[
\text{the most negative point is the center of the fundamental alcove } A. \tag{1.11}
\]

Using the orientation of the walls given by (3.11) produces the basis commonly denoted \( \{T_w \mid w \in \tilde{W}\} \) by taking \( T_w \) to be the image in \( \tilde{H} \) of a minimal length alcove walk from \( A \) to \( w^{-1}A \). Since \( T_i^{-1} = T_i - (q - q^{-1}) \) the transition matrix between the basis \( \{X^\lambda T_{w^{-1}}^{-1} \mid \lambda \in P, w \in W\} \) and the basis \( \{T_w \mid w \in \tilde{W}\} \) is triangular.