\[ \pi \] is the distance halfway around a circle of radius 1. Measure angles according to the distance traveled on a circle of radius 1.

\[ \text{The angle } \theta \text{ is measured by traveling a distance } \theta \text{ on a circle of radius 1.} \]

\[ \text{Stretch both } x \text{ and } y \text{ to get a circle of radius } r. \]

\[ \text{The distance } \theta \text{ stretches to } r\theta. \text{ Hence, the} \]

\[ (\text{arc length along an angle } \theta \text{ on a circle of radius } r) = r\theta. \]

The distance \(2\pi\) around a circle of radius 1 stretches to \(2\pi r\) around a circle of radius \(r\). So the circumference of a circle is \(2\pi r\) is the circle is radius \(r\).

To find the area of a circle first approximate with a polygon inscribed in the circle.

\[ \text{the eight triangles form an octagon } P_8 \text{ in the circle. The area of the octagon is almost the same as the area of the circle.} \]

Unwrap the octagon.

\[ \text{The area of the octagon is the area of the 8 triangles. The area of each triangle is } \frac{1}{2}bh. \text{ So the area of the octagon is } \frac{1}{2}Bh. \]
Take the limit as the number of triangles in the interior polygon gets larger and larger (the polygon gets closer and closer to being the circle). Then

\[
\text{Area of the circle} = \lim_{n \to \infty} \left( \text{area of an } n\text{-sided polygon } P_n \right)
= \lim_{n \to \infty} \left( \frac{1}{2} Bh \right)
\]

\begin{align*}
\text{PICTURE} & \text{total base height of triangle} \\
& = \frac{1}{2} (2\pi r) (r)
\end{align*}

\begin{align*}
\text{PICTURE} & \text{length of an unwrapped circle radius of the circle} \\
& = \pi r^2.
\end{align*}

So the area of a circle is \( \pi r^2 \) if the circle is radius \( r \), and the

\[
(\text{area of an arc of angle } \theta \text{ for a circle of radius } r) = \frac{\theta}{2\pi} \cdot (\text{area of the whole circle})
= \frac{\theta}{2\pi} \cdot \pi r^2 = \frac{\theta r^2}{2}.
\]

**Trigonometric functions**

\[
\sin \theta \text{ is the } y\text{-coordinate of a point at distance } \theta \text{ on a circle of radius 1,}
\cos \theta \text{ is the } x\text{-coordinate of a point at distance } \theta \text{ on a circle of radius 1,}
\]

\[
\tan \theta = \frac{\sin \theta}{\cos \theta},
\cot \theta = \frac{\cos \theta}{\sin \theta},
\sec \theta = \frac{1}{\cos \theta},
\csc \theta = \frac{1}{\sin \theta}.
\]

Since the equation of a circle of radius 1 is \( x^2 + y^2 = 1 \) this forces

\[
\sin^2 \theta + \cos^2 \theta = 1.
\]

The pictures

\begin{align*}
\text{PICTURE} & \text{ and } \text{PICTURE}
\end{align*}

show that

\[
\sin(-\theta) = -\sin \theta \quad \text{and} \quad \cos(-\theta) = \cos \theta.
\]

Also

\begin{align*}
\text{PICTURE} & \text{ and } \text{PICTURE}
\end{align*}

show that

\[
\sin 0 = 0 \quad \text{and} \quad \sin \frac{\pi}{2} = 1,
\]

\[
\cos 0 = 1 \quad \text{and} \quad \cos \frac{\pi}{2} = 0.
\]
Draw the graphs $\text{PICTURE}$ and $\text{PICTURE}$, by seeing how the $x$ and $y$ coordinates change as you walk around the circle.

There are five trig identities to remember:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y,$$
$$\sin^2 x + \cos^2 x = 1,$$
$$\sin(-x) = -\sin x \quad \text{and} \quad \cos(-x) = \cos x,$$

As well as the two triangles $\text{PICTURE}$ and $\text{PICTURE}$.

From these triangles,

$$\sin \frac{\pi}{6} = \frac{1}{2} \quad \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2},$$
$$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2},$$
$$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \quad \cos \frac{\pi}{3} = \frac{1}{2}.$$  

Since the only trig identities I remember are identities for sines and cosines I usually verify trig identities by first writing them completely in terms of sines and cosines.

Example. Verify $\frac{\sec B}{\cos B} - \frac{\tan B}{\cot B} = 1$.

$$= \frac{1}{\cos^2 B} - \frac{\sin^2 B}{\cos^2 B} = 1 - \frac{\sin^2 B}{\cos^2 B} = \frac{\cos^2 B}{\cos^2 B} = 1.$$  

Example. Verify $\frac{\cot \alpha - \cot \beta}{\sin \alpha \sin \beta} = \sin(\beta - \alpha)$.

Left Hand Side = $\frac{\cos \alpha}{\sin \alpha} - \frac{\cos \beta}{\sin \beta}$

$$= \frac{\cos \alpha \sin \beta - \cos \beta \sin \alpha}{\sin \alpha \sin \beta}$$

Right Hand Side = $\frac{\sin(\beta - \alpha)}{\sin \alpha \sin \beta} = \frac{\sin \beta \cos(-\alpha) + \cos \beta \sin(-\alpha)}{\sin \alpha \sin \beta}$

$$= \frac{\sin \beta \cos \alpha + \cos \beta (-\sin \alpha)}{\sin \alpha \sin \beta} = \frac{\sin \beta \cos \alpha - \cos \beta \sin \alpha}{\sin \alpha \sin \beta}.$$  

So

Left Hand Side = Right Hand Side.
Example. Verify \( \frac{\tan A - \sin A}{\sec A} = \frac{\sin^3 A}{1 + \cos A} \).

\[
\tan A - \sin A \quad \frac{\tan A}{\sec A} = ? \quad \frac{\sin^3 A}{1 + \cos A}
\]

So \((1 + \cos A)(\tan A - \sin A) = ? \sin^3 A \sec A\).

So \(\tan A + \cos A \tan A - \sin A - \sin A \cos A = ? \sin^3 A \sec A\).

So \(\sin A - \sin \cos A + \cos A = ? \sin^3 A \left( \frac{1}{\cos A} \right) \).

So \(\sin A - \sin \cos A = ? \sin^3 A \left( \frac{1}{\cos A} \right) \).

So \(1 - \cos^2 A = ? \sin^2 A\).

YES, because \(\sin^2 A + \cos^2 A = 1\).

and so

\[
\text{adam}(t) = x\text{-coordinate of the point on a circle of radius 1 which is distance } d \text{ from the point (1,0)}, \quad \text{and}
\]

\[
\text{eve}(t) = y\text{-coordinate of the point on a circle of radius 1 which is distance } d \text{ from the point (1,0)}.\]
The triangle in this picture is

\[
\begin{array}{c}
\text{1} \\
\text{adam}(d) \\
\hline
\text{eve}(d) \\
\text{hypotenuse} \\
\text{opposite} \\
\text{adjacent}
\end{array}
\]

and so

\[
\text{adam}(d) = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{and} \quad \text{eve}(d) = \frac{\text{adjacent}}{\text{hypotenuse}}
\]

for a right triangle with angle \(d\).