1. Algebras and representations.

An algebra is a vector space (over \( \mathbb{C} \)) with a multiplication such that \( A \) is a ring with identity, i.e. there is a map \( A \times A \rightarrow A, \ (a, b) \mapsto ab \), which is bilinear and satisfies the associative and distributive laws. The following are examples of algebras:

1. The group algebra of a group \( G \) is the vector space \( \mathbb{C}G \) with basis \( G \) and with multiplication forced by the multiplication in \( G \) (and the bilinearity).
2. If \( M \) is a vector space (over \( \mathbb{C} \)) then the space \( \text{End}(M) \) of \( \mathbb{C} \)-linear transformations of \( M \) is an algebra under the multiplication given by composition of endomorphisms.
3. Given a basis \( B = \{b_1, \ldots, b_d\} \) of the vector space \( M \) the algebra \( \text{End}(M) \) can be identified with the algebra \( M_d(\mathbb{C}) \) of \( d \times d \) matrices \( T = (T_{ij})_{1 \leq i, j \leq d} \) with entries in \( \mathbb{C} \) via

\[
Tb_i = \sum_{j=1}^{d} b_j T_{ji}, \quad \text{for } t \in \text{End}(M).
\]

Let \( A \) be an algebra. An ideal in \( A \) is a subspace \( I \subset A \) such that \( ar \in I \) and \( ra \in I \), for all \( a \in A \) and \( r \in I \). A minimal ideal of \( A \) is a nonzero ideal \( I \) which cannot be written as a direct sum \( I = I_1 \oplus I_2 \) of nonzero ideals \( I_1 \) and \( I_2 \) of \( A \). An idempotent is a nonzero element \( p \in A \) such that \( p^2 = p \). Two idempotents \( p_1, p_2 \in A \) are orthogonal if \( p_1 p_2 = p_2 p_1 = 0 \). A minimal idempotent is an idempotent \( p \) that cannot be written as a sum \( p = p_1 + p_2 \) of orthogonal idempotents \( p_1, p_2 \in A \). The center of \( A \) is

\[
Z(A) = \{z \in A \mid az = za \text{ for all } a \in A\}.
\]

A central idempotent is an idempotent in \( Z(A) \) and a minimal central idempotent is a central idempotent \( z \) that cannot be written as a sum \( z = z_1 + z_2 \) of orthogonal central idempotents \( z_1 \) and \( z_2 \).

A trace on \( A \) is a linear map \( \tilde{t}: A \rightarrow \mathbb{C} \) such that

\[
\tilde{t}(a_1 a_2) = \tilde{t}(a_2 a_1), \quad \text{for all } a_1, a_2 \in A.
\]
A character of $A$ is a trace on $A$. A trace $\bar{t}$ on $A$ is nondegenerate if for each $b \in A$ there is an $a \in A$ such that $\bar{t}(ba) \neq 0$. The radical of a trace $\bar{t}$ is

$$\text{rad } t = \{ b \in A \mid \bar{t}(ba) = 0 \text{ for all } a \in A \}. \quad (1.1)$$

Every trace $\bar{t}$ on $A$ determines a symmetric bilinear form $\langle \cdot , \cdot \rangle : A \times A \to \mathbb{C}$ given by

$$\langle a_1, a_2 \rangle = \bar{t}(a_1a_2), \quad \text{for all } a_1, a_2 \in A. \quad (1.2)$$

The form $\langle \cdot , \cdot \rangle$ is nondegenerate if and only if the trace $\bar{t}$ is nondegenerate and the radical

$$\text{rad } \langle \cdot , \cdot \rangle = \{ b \in A \mid \langle a, b \rangle = 0 \text{ for all } a \in A \}$$

of the form $\langle \cdot , \cdot \rangle$ is the same as $\text{rad } \bar{t}$.

**Lemma 1.3.** Let $\bar{t}$ be a trace on $A$ and let $\langle \cdot , \cdot \rangle$ be the bilinear form on $A$ defined by the trace $\bar{t}$, as in ???. Let $B$ be a basis of $A$. Let $G = (\langle b, b' \rangle)_{b, b' \in B}$ be the matrix of the form $\langle \cdot , \cdot \rangle$ with respect to $B$. The following are equivalent:

1. The trace $\bar{t}$ is nondegenerate.
2. $\det G \neq 0$.
3. The dual basis $B^*$ to the basis $B$ with respect to the form $\langle \cdot , \cdot \rangle$ exists.

**Proof.** (2) $\Leftrightarrow$ (1): The trace $\bar{t}$ is degenerate if there is an element $a \in A$, $a \neq 0$, such that $\bar{t}(ac) = 0$ for all $c \in B$. If $a_b \in \mathbb{C}$ are such that

$$a = \sum_{b \in B} a_b b, \quad \text{then} \quad 0 = \langle a, c \rangle = \sum_{b \in B} a_b \langle b, c \rangle$$

for all $c \in B$. So $a$ exists if and only if the columns of $G$ are linearly dependent, i.e. if an only if $G$ is not invertible.

(3) $\Leftrightarrow$ (2): Let $B^* = \{ b^* \}$ be the dual basis to $\{ b \}$ with respect to $\langle \cdot , \cdot \rangle$ and let $P$ be the change of basis matrix from $B$ to $B^*$. Then

$$d^* = \sum_{b \in B} P_{db} b, \quad \text{and} \quad \delta_{bc} = \langle b, d^* \rangle = \sum_{d \in B} P_{dc} \langle b, c \rangle = (GP^t)_{b, c}.$$

So $P^t$, the transpose of $P$, is the inverse of the matrix $G$. So the dual basis to $B$ exists if and only if $G$ is invertible, i.e. if and only if $\det G \neq 0$. $\blacksquare$

**Proposition 1.4.** Let $A$ be an algebra and let $\bar{t}$ be a nondegenerate trace on $A$. Define a symmetric bilinear form $\langle \cdot , \cdot \rangle : A \times A \to \mathbb{C}$ on $A$ by $\langle a_1, a_2 \rangle = \bar{t}(a_1a_2)$, for all $a_1, a_2 \in A$. Let $B$ be a basis of $A$ and let $B^*$ be the dual basis to $B$ with respect to $\langle \cdot , \cdot \rangle$. Let $a \in A$ and define

$$[a] = \sum_{b \in B} bab^*.$$

Then $[a]$ is an element of the center $Z(A)$ of $A$ and $[a]$ does not depend on the choice of the basis $B$. 

Proof. Let $c \in A$. Then

$$c[a] = \sum_{b \in B} \sum_{d \in B} \langle cb, d^* \rangle dab^* = \sum_{d \in B} da \sum_{b \in B} \langle d^* c, b \rangle b^* = \sum_{d \in B} d\!\cdot\! a^* c = [a]c,$$

since $\langle cb, d^* \rangle = \tilde{t}(cbd^*) = \tilde{t}(d^* cb) = \langle d^* c, b \rangle$. So $[a] \in Z(A)$.

Let $D$ be another basis of $A$ and let $D^*$ be the dual basis to $D$ with respect to $\langle , \rangle$. Let $P = (P_{db})$ be the transition matrix from $D$ to $B$ and let $P^{-1}$ be the inverse of $P$. Then

$$d = \sum_{b \in B} P_{db} b \quad \text{and} \quad d^* = \sum_{b \in B} (P^{-1})_{bd} b^*,$$

since

$$\langle d, d^* \rangle = \left\langle \sum_{b \in B} P_{db} b, \sum_{b \in B} (P^{-1})_{bd} b^* \right\rangle = \sum_{b, b \in B} P_{db} (P^{-1})_{bd} \delta_{bb} = \delta_{dd}.$$

So

$$\sum_{d \in D} d a d^* = \sum_{d \in D} \sum_{b \in B} P_{db} a b \sum_{b} (P^{-1})_{bd} b^* = \sum_{b, b \in B} bab^* \delta_{bb} = \sum_{b} bab^*.$$

So $[a]$ does not depend on the choice of the basis $B$. 

Representations.

An $A$-module is a vector space $M$ (over $\mathbb{C}$) with an $A$-action, i.e. a map $A \times M \to M$, $(a, m) \mapsto am$, which is bilinear and such that

$$1_A m = m \quad \text{and} \quad a_1 (a_2 m) = (a_1 a_2) m,$$

for all $a_1, a_2 \in A$ and $m \in M$ ($1_A$ denotes the identity in the algebra $A$). A representation of $A$ is an $A$-module. A representation of a group $G$ is a representation of the group algebra $\mathbb{C}G$. The character of an $A$-module $M$ is the map $\chi^M : A \to \mathbb{C}$ given by

$$\chi^M(a) = \text{Tr}(M(a)), \quad \text{for } a \in A,$$

where $M(a)$ is the linear transformation of $M$ determined by the action of $A$ and $\text{Tr}(M(a))$ is the trace of $M(a)$. An irreducible character of $A$ is the character of an irreducible representation of $A$.

An $A$-module $M$ gives rise to a map

$$A \to \text{End}(M), \quad a \mapsto M(a) \quad (1.5)$$

where $M(a)$ is the linear transformation of $M$ determined by the action of $a$ on $M$. This map is linear and satisfies

$$M(1_A) = \text{Id}_M,$$

$$M(a_1 a_2) = M(a_1) M(a_2),$$

for all $a_1, a_2 \in A$, i.e. $A \to \text{End}(M)$ is a homomorphism of algebras. (Given a basis $B = \{b_1, \ldots, b_d\}$ of $M$ the map $A \to \text{End}(M)$ can be identified with a map $M : A \to M_d(\mathbb{C})$.) Conversely, an algebra homomorphism as in (1.4) and (1.5) determines an $A$-action on $M$ by

$$am = M(a)m, \quad \text{for all } a \in A \text{ and } m \in M.$$
Thus, the map $M: A \to \text{End}(M)$ and the $A$-module $M$ are equivalent data. Historically, the map $M: A \to \text{End}(M)$ was the representation and $M$ was the $A$-module, but now the terms representation and $A$-module are used interchangeably. This is the reason for the use of the letter $M$, both for the $A$-module and the corresponding algebra homomorphism $M: A \to \text{End}(M)$.

A submodule of an $A$-module $M$ is a subspace $N \subseteq M$ such that $an \in N$, for all $a \in A$ and $n \in N$. An $A$-module $M$ is simple or irreducible if it has no submodules except 0 and itself. The direct sum of two $A$-modules $M_1$ and $M_2$ is the vector space $M = M_1 \oplus M_2$ with $A$-action given by

\[ a(m_1, m_2) = (am_1, am_2), \quad \text{for all } a \in A, m_1 \in M_1 \text{ and } m_2 \in M_2. \]

An $A$-module $M$ is semisimple or completely decomposable if $M$ can be written as a direct sum of simple submodules. An $A$-module $M$ is indecomposable if $M$ cannot be written as a direct sum $M = M_1 \oplus M_2$ of nonzero submodules $M_1 \subseteq M$ and $M_2 \subseteq M$.

Here we need a reference to the reader to look at the examples in Chapter 2 etc.

**Proposition 1.7.** Let $A$ and $B$ be algebras and let $A^\lambda$, $\lambda \in \hat{A}$, and $B^\mu$, $\mu \in \hat{B}$, be the irreducible representations of $A$ and $B$, respectively. The irreducible representations of $A \oplus B$ are $A^\lambda$, $\lambda \in \hat{A}$, with $A \oplus B$ action given by

\[ (a, b)m = am, \quad \text{for } a \in A, b \in B, m \in A^\lambda, \]

and $B^\mu$, $\mu \in \hat{B}$, with $A \oplus B$ action given by

\[ (a, b)n = bn, \quad \text{for } a \in A, b \in B, \text{ and } n \in B^\mu. \]

**Proof.** The elements $(1, 0)$ and $(0, 1)$ in $A \oplus B$ are central idempotents of $A \oplus B$ such that $(1, 0)(0, 1) = (0, 0)$. If $P$ is an $A \oplus B$-module then

\[ P = (1, 0)P \oplus (0, 1)P, \]
and this is a decomposition as \( A \oplus B \)-modules. Since
\[
(a, b)(1, 0)p = (a, 0)(1, 0)p, \quad \text{and} \quad (a, b)(0, 1)p = (0, b)(0, 1)p,
\]
for all \( a \in A, b \in B, \) and \( p \in P, \) the structure of \((1, 0)P\) is determined completely by the \( A\)-action and the structure of \((0, 1)P\) is determined by the action of \( B. \) If \( P \) is a simple module then \( P = (1, 0)P \) or \( P = (0, 1)P. \) In the first case \( P \cong A^\lambda \) for some \( \lambda \in \hat{A} \) and in the second \( P \cong B^\mu \) for some \( \mu \in \hat{B}. \)

Similar arguments with the elements \((1, 0)\) and \((0, 1)\) in \( A \oplus B \) yield the following.

1. If \( A \) and \( B \) are algebras then the ideals of \( A \oplus B \) are all of the form \( I \oplus J \) where \( I \) is an ideal of \( A \) and \( J \) is an ideal of \( B. \)
2. If \( A \) and \( B \) are algebras then \( Z(A \oplus B) = Z(A) \oplus Z(B). \)
3. If \( A \) and \( B \) are algebras and \( \tilde{t} \) is a trace on \( A \oplus B \) then \( \tilde{t} \) is given by
\[
\tilde{t}(a, b) = \tilde{t}_A(a) + \tilde{t}_B(b),
\]
where \( \tilde{t}_A \) is the trace on \( A \) given by \( \tilde{t}_A(a) = \tilde{t}(a, 0) \) and \( \tilde{t}_B \) is the trace on \( B \) given by \( \tilde{t}_B(b) = \tilde{t}(0, b). \)

**Tensor products**

Let \( M \) and \( N \) be vector spaces and let
\[
B_m = \{m_i\} \quad \text{and} \quad B_n = \{n_j\}
\]
be bases of \( M \) and \( N, \) respectively. The tensor product \( M \otimes N \) is the vector space with basis
\[
B_{M \otimes N} = \{m_i \otimes n_j \mid m_i \in B_M, n_j \in B_N\}.
\]
If \( m = \sum_i c_i m_i, \) and \( n = \sum_j d_j n_j, \) then write
\[
m \otimes n = \left( \sum_i c_i m_i \right) \otimes \left( \sum_j d_j n_j \right) = \sum_{i,j} c_i d_j (m_i \otimes n_j).
\]

If \( A \) and \( Z \) are algebras the **tensor product** is the vector space \( A \otimes Z \) with multiplication determined by
\[
(a_1 \otimes z_1)(a_2 \otimes z_2) = a_1 a_2 \otimes z_1 z_2, \quad \text{for all} \ a_1, a_2 \in A, \ z_1, z_2 \in Z.
\]
If \( M \) and \( N \) are vector spaces then
\[
\text{End}(M \otimes N) = \text{End}(M) \otimes \text{End}(N) \quad \text{as algebras.}
\]
This equality can be expressed in terms of matrices by choosing bases \( \{m_1, \ldots, m_r\} \) and \( \{n_1, \ldots, n_s\} \) of \( M \) and \( N, \) respectively. The \( \text{End}(M) \) is identified with \( M_r(\mathbb{C}) \) and \( \text{End}(N) \) is identified with \( M_s(\mathbb{C}) \) by
\[
E_{ij}m_j = m_i \quad \text{and} \quad E_{kl}n_k = n_k, \quad \text{for} \ 1 \leq i, j \leq r \ \text{and} \ 1 \leq k, \ell \leq s.
\]
Then
\[(E_{ij} \otimes E_{k\ell})(m_j \otimes n_\ell) = E_{ij}m_j \otimes E_{k\ell}n_\ell = m_i \otimes n_k.\]

Use the (ordered) basis
\[\{m_1 \otimes n_1, \ldots, m_1 \otimes n_s, m_2 \otimes n_1, \ldots, m_2 \otimes n_s, \ldots, m_r \otimes n_1, \ldots, m_r \otimes n_s\}\]
of \(M \otimes N\) to identify \(\text{End}(M \otimes N)\) with \(M_{rs}(\mathbb{C})\). Then, if \(a = (a_{ij}) \in M_r(\mathbb{C})\) and \(b = (b_{k\ell}) \in M_s(\mathbb{C})\) then \(a \otimes b\) is the \(rs \times rs\) matrix
\[
a \otimes b = \begin{pmatrix}
a_{11}b & a_{12}b & \cdots & a_{1r}b \\
a_{21}b & a_{22}b & \cdots & a_{2r}b \\
\vdots & \ddots & \ddots & \vdots \\
a_{r1}b & a_{r2}b & \cdots & a_{rr}b
\end{pmatrix}
\]

**Theorem 1.8.** Let \(A\) and \(B\) be algebras. Let \(A^\lambda, \lambda \in \hat{A}\), be the simple \(A\)-modules and let \(B^\mu, \mu \in \hat{B}\), be the simple \(B\)-modules. The simple \(A \otimes B\)-modules are
\[A^\lambda \otimes B^\mu, \quad \lambda \in \hat{A}, \mu \in \hat{B}, \quad \text{where} \quad (a \otimes b)(m \otimes n) = am \otimes bn,
\]
for \(a \in A, b \in B, m \in A^\lambda, n \in B^\mu\).

**Proof.** There are two things to show:

1. \(A^\lambda \otimes B^\mu\) is a simple \(A \otimes B\)-module,
2. If \(P\) is a simple \(A \otimes B\)-module then \(P \cong A^\lambda \otimes B^\mu\) for some \(\lambda \in \hat{A}\) and \(\mu \in \hat{B}\).

(1) By Burnside’s theorem \(\text{End}(A^\lambda) = A^\lambda(A)\) and \(\text{End}(B^\mu) = B^\mu(B)\) and therefore
\[\text{End}(A^\lambda \otimes B^\mu) = \text{End}(A^\lambda) \otimes \text{End}(B^\mu) = A^\lambda(A) \otimes B^\mu(B) = (A^\lambda \otimes B^\mu)(A \otimes B).\]

So \(A^\lambda \otimes B^\mu\) has no submodules. So \(A^\lambda \otimes B^\mu\) is simple.

(2) Let \(P\) be a simple \((A \otimes B)\)-module. Let \(A^\lambda\) be a simple \(A\)-submodule of \(P\) and let \(B^\mu\) be a simple \(B\)-submodule of \(\text{Hom}_A(A^\lambda, P)\). We claim that \(A^\lambda \otimes B^\mu \cong P\).

Consider the \((A \otimes B)\)-module homomorphism
\[
\Phi: A^\lambda \otimes B^\mu \rightarrow A^\lambda \otimes \text{Hom}_A(A^\lambda, P) \rightarrow P
\]
\[
m \otimes \phi \mapsto \phi(m).
\]
This map is nonzero since the injection \(\phi: A^\lambda \rightarrow P\) is a nonzero element of \(\text{Hom}_A(A^\lambda, P)\). Since \(A^\lambda \otimes B^\mu\) is simple ker \(\Phi = 0\) and since \(P\) is simple \(\text{im}\Phi = P\). So \(A^\lambda \otimes B^\mu \cong P\). 

**2. The algebra \(M_d(\mathbb{C})\).**

Let \(A = M_d(\mathbb{C})\) be the algebra of \(d \times d\) matrices with entries from \(\mathbb{C}\). Set
\[E_{ij} = \text{the matrix with 1 in the } (i, j) \text{ entry and all other entries 0.}\]

Then \(\{E_{ij} \mid 1 \leq i, j \leq d\}\) is a basis of \(A\) and
\[E_{ij}E_{kl} = \delta_{jk}E_{il}, \quad 1 \leq i, j, k, l \leq d,
\]
describes the multiplication in \( A \).

**Theorem 2.1.** Let \( M_d(\mathbb{C}) \) be the algebra of \( d \times d \) matrices with entries from \( \mathbb{C} \).

(a) Up to isomorphism, there is only one irreducible representation \( M \) of \( M_d(\mathbb{C}) \).

(b) \( \dim(M) = d \).

(c) The character \( \chi^M : A \to \mathbb{C} \) of \( M \) is given by

\[
\chi^M(a) = \text{Tr}(a), \quad \text{for all } a \in A,
\]

where \( \text{Tr}(a) \) is the trace of the matrix \( a \).

(d) The irreducible representation \( M \) is the vector space

\[
M = \{ (c_1, \ldots, c_d)^t \mid c_i \in \mathbb{C} \}
\]

of column vectors of length \( d \) with \( A \)-action given by left multiplication, or, equivalently, \( M \) is given by the map

\[
M : A \to M_d(\mathbb{C}), \quad a \mapsto (a).
\]

**Proof.** There are two things to show:

1. \( M \), as defined in (d), is a simple \( A \)-module, and
2. If \( C \) is a simple \( A \)-module then \( C \cong M \).

(1) Let \( \epsilon_i \) be the column vector which has 1 in the \( i \)th entry and 0 in all other entries. The set \( \{ \epsilon_1, \ldots, \epsilon_d \} \) is a basis of \( M \). Let \( N \subseteq M \) be a nonzero submodule of \( M \) and let \( n = \sum_{i=1}^{d} n_i \epsilon_i \) be a nonzero vector in \( N \). Then \( n_j \neq 0 \) for some \( j \) and so

\[
\epsilon_k = \frac{1}{n_j} E_{kj} n \in N, \quad \text{for all } 1 \leq k \leq d.
\]

Thus \( N = M \), since \( N \) contains a basis of \( M \).

(2) Let \( C \) be a simple \( A \)-module and let \( c \) be a nonzero vector in \( C \). Since \( c = \text{Id} \cdot c = \sum_{i=1}^{d} E_{ii} c \neq 0 \), \( E_{jj} c \neq 0 \) for some \( j \). Define an \( A \)-module homomorphism by

\[
\phi : M \to C, \quad \epsilon_k \mapsto E_{kj} c.
\]

Since \( \phi(\epsilon_j) \neq 0, \ker \phi \neq 0 \). Since \( M \) is simple, \( \ker \phi = M \) and so \( \phi \) is injective. Since \( \text{im} \phi \neq 0 \) and \( C \) is simple, \( \text{im} \phi = C \) and so \( \phi \) is surjective. So \( \phi \) is an isomorphism and \( C \cong M \). □

**Proposition 2.2.** Let \( M_d(\mathbb{C}) \) be the algebra of \( d \times d \) matrices with entries from \( \mathbb{C} \).

1. The only ideals of \( M_d(\mathbb{C}) \) are 0 and \( M_d(\mathbb{C}) \).
2. \( Z(M_d(\mathbb{C})) = \mathbb{C} \cdot \text{Id} \) and \( \text{Id} \) is the only central idempotent in \( M_d(\mathbb{C}) \).
3. Up to constant multiples, the trace \( \text{Tr} : M_d(\mathbb{C}) \to \mathbb{C} \) given by

\[
\text{Tr}(a) = \sum_{i=1}^{d} a_{ii}, \quad \text{for all } a = (a_{ij}) \in M_d(\mathbb{C}),
\]
is the unique trace on $M_d(\mathbb{C})$.

**Proof.** Let $E_{ij}$ denote the matrix in $M_d(\mathbb{C})$ which has a 1 in the $(i, j)$ entry and 0 everywhere else.

1. Let $I$ be a nonzero ideal of $M_d(\mathbb{C})$ and let $r = (r_{ij}) \in I$, $r \neq 0$. Let $r_{ij}$ be a nonzero entry of $r$. Then

$$\frac{1}{r_{ij}} E_{ki} r E_{jl} = E_{kl} \in I, \quad \text{for all } 1 \leq k, l \leq d.$$ 

So $I$ contains a basis of $M_d(\mathbb{C})$. So $I = M_d(\mathbb{C})$.

2. Clearly $\mathbb{C}I_d \subseteq Z(M_d(\mathbb{C})$. Let $z = (z_{ij}) \in Z(M_d(\mathbb{C}))$. If $i \neq j$ then

$$z_{ij} E_{ij} = E_{ii} z_{jj} = \delta_{ij} \chi(E_{11}),$$

so $z_{ij} = 0$ if $i \neq j$. Further

$$z_{ii} E_{ii} = E_{ii} z_{ii} = E_{ii} z_{ii} = z_{ii} E_{ii},$$

so $z_{ii} = z_{11}$ for all $1 \leq i \leq d$. So $z = z_{11} I_d$. So $Z(M_d(\mathbb{C})) \subseteq \mathbb{C}I_d$. So $Z(M_d(\mathbb{C})) = \mathbb{C}I_d$.

3. Let $\chi: M_d(\mathbb{C}) \to \mathbb{C}$ be a trace on $M_d(\mathbb{C})$. If $a = (a_{ij}) \in M_d(\mathbb{C})$ then

$$\chi(E_{ij} a E_{jj}) = a_{ij} \chi(E_{ij}) = a_{ij} \chi(E_{1j} E_{j1}) = a_{ij} \delta_{ij} \chi(E_{11}).$$

Thus

$$\chi(a) = \chi \left( \left( \sum_{i=1}^{d} E_{ii} \right) a \left( \sum_{j=1}^{d} E_{jj} \right) \right) = \sum_{i,j=1}^{d} a_{ij} \delta_{ij} \chi(E_{11}) = \chi(E_{11}) \text{Tr}(a).$$

So $\chi$ is a multiple of the trace Tr. 

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3. **The algebra $\bigoplus_{\lambda \in \hat{\lambda}} M_{d_{\lambda}}(\mathbb{C})$.**

Let $\hat{\lambda}$ be a finite set and let $d_{\lambda}$ be positive integers indexed by the elements of $\hat{\lambda}$. Let

$$A = \bigoplus_{\lambda \in \hat{\lambda}} M_{d_{\lambda}}(\mathbb{C}),$$

be the algebra of block diagonal matrices with blocks $M_{d_{\lambda}}(\mathbb{C})$. Let $E_{ij}^\lambda$ be the matrix which has a 1 in the $(i, j)$ entry of the $\lambda$th block and 0 everywhere else. Then $\{E_{ij}^\lambda \mid \lambda \in \hat{\lambda}, 1 \leq i, j \leq d_{\lambda}\}$ is a basis of $A$ and the relations

$$E_{ij}^\lambda E_{kl}^\mu = \delta_{\lambda\mu} \delta_{ij} E_{kl}^\lambda$$

determine the multiplication in $A$. 

The following theorems are consequences of Theorems ?? and Proposition ???.

**Theorem 3.1.** Let $\hat{\lambda}$ be a finite set and let $d_{\lambda}$ be positive integers indexed by the elements of $\hat{\lambda}$. Let

$$A = \bigoplus_{\lambda \in \hat{\lambda}} M_{d_{\lambda}}(\mathbb{C}),$$

be the algebra of block diagonal matrices with blocks $M_{d_{\lambda}}(\mathbb{C})$. 

(1) The irreducible representations $A^\lambda$ of $A$ are indexed by the elements of $\hat{A}$.

(2) $\dim(A^\lambda) = d_\lambda$.

(3) The character $\chi^\lambda: A \to \mathbb{C}$ of $A^\lambda$ is given by
$$\chi^\lambda(a) = \text{Tr}(A^\lambda(a)), \quad a \in A,$$
where $A^\lambda(a)$ is the $\lambda$th block of the matrix $a$.

(4) The irreducible representation $A^\lambda$ is given by the map
$$A^\lambda: A \longrightarrow M_{d_\lambda}(\mathbb{C}),$$
where $A^\lambda(a)$ is the $\lambda$th block of the matrix $A$, or, equivalently, by the vector space $A^\lambda$ of column vectors of length $d_\lambda$ and $A$-action given by
$$am = A^\lambda(a)m, \quad \text{for } a \in A \text{ and } m \in A^\lambda.$$

**Theorem 3.2.** Let $\hat{A}$ be a finite set and let $d_\lambda$ be positive integers indexed by the elements of $\hat{A}$. Let
$$A = \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}),$$
be the algebra of block diagonal matrices with blocks $M_{d_\lambda}(\mathbb{C})$. If $a \in A$ let $A^\lambda(a)$ denote the $\lambda$th block of the matrix $a$. Let $E_{ij}^\lambda$ be the matrix which has a 1 in the $(i,j)$ entry of the $\lambda$th block and 0 everywhere else.

(1) The minimal ideals of $A$ are given by
$$I^\lambda = \{a \in A \mid A^\mu(a) = 0 \text{ for all } \mu \neq \lambda\}, \quad \lambda \in \hat{A},$$
and every ideal of $A$ is of the form $I = \bigoplus_{\lambda \in S} I^\lambda$, for some subset $S \subseteq \hat{A}$.

(2) The minimal central idempotents of $A$ are
$$z_\lambda = \sum_{i=1}^{d_\lambda} E_{ii}^\lambda, \quad \lambda \in \hat{A},$$
and $\{z_\lambda \mid \lambda \in \hat{A}\}$ is a basis of the center $Z(A)$ of $A$.

(3) The irreducible characters $\chi^\lambda$, $\lambda \in \hat{A}$, of $A$ are given by
$$\chi^\lambda(a) = \text{Tr}(A^\lambda(a)), \quad a \in A,$$
and every trace $\bar{t}: A \to \mathbb{C}$ on $A$ can be written uniquely in the form
$$\bar{t} = \sum_{\lambda \in \hat{A}} t_\lambda \chi^\lambda, \quad t_\lambda \in \mathbb{C}.$$
Let $A$ be an algebra which is isomorphic to a direct sum of matrix algebras and fix an isomorphism
\[ \phi: A \cong \bigoplus_{\lambda \in \hat{A}} M_{d_{\lambda}}(\mathbb{C}). \]  
(3.3)

The elements
\[ e_{ij}^\lambda = \phi^{-1}(E_{ij})^\lambda, \quad \lambda \in \hat{A}, \quad 1 \leq i, j \leq d_{\lambda}, \]
are matrix units in $A$, i.e. \{\(e_{ij}^\lambda | \lambda \in \hat{A}, 1 \leq i, j \leq d_{\lambda}\)\} is a basis of $A$ and
\[ e_{ij}^\lambda e_{kl}^\mu = \delta_{\lambda\mu} \delta_{ij} e_{il}^\lambda, \]
for all $\lambda, \mu \in \hat{A}$, $1 \leq i, j \leq d_{\lambda}$, $1 \leq k, l \leq d_{\mu}$. If $a \in A$, let $A^\lambda(a)_{ij} \in \mathbb{C}$ be defined by the expansion
\[ a = \sum_{\lambda \in \hat{A}} \sum_{i,j=1}^{d_{\lambda}} A^\lambda(a)_{ij} e_{ij}^\lambda. \]

It follows from Theorem ??? that the maps
\[ A^\lambda: A \to M_{d_{\lambda}}(\mathbb{C}) \quad \text{and} \quad \chi^\lambda: A \to \mathbb{C} \quad \text{are the irreducible representations and the irreducible characters of} \ A, \text{respectively. The homomorphisms} \ A^\lambda \text{depend on the choice of} \ \phi \text{but the irreducible characters} \ \chi^\lambda \text{do not. The weights of a trace} \ \tilde{t} \text{on} \ A \text{are the constants} \ t_{\lambda}, \ \lambda \in \hat{A}, \text{defined by the expansion in} \ ??? \text{. The trace} \ \tilde{t} \text{is nondegenerate if and only if the} \ t_{\lambda} \text{are all nonzero.}

**Theorem 3.4.** Let $A$ be an algebra which is isomorphic to a direct sum of matrix algebras, indexed by $\lambda \in \hat{A}$. Let $\tilde{t}$ be a nondegenerate trace on $A$ and let $\langle \cdot, \cdot \rangle$ be the corresponding bilinear form. Let $B = \{b\}$ be a basis of $A$ and let $B^* = \{b^*\}$ be the dual basis to $B$ with respect to $\langle \cdot, \cdot \rangle$. Let $\chi^\lambda, \lambda \in \hat{A}$, be the irreducible characters of $A$, $t_{\lambda}$ be the weights of $\tilde{t}$, $d_{\lambda}$ the dimensions of the irreducible representations, $\{e_{ij}^\lambda\}$ a set of matrix units of $A$, and $A^\lambda$ the corresponding irreducible representations of $A$.

(a) (Fourier inversion formula)
\[ e_{ij}^\lambda = \sum_{b \in B} t_{\lambda} A_{ji}^\lambda (b^*) b. \]

(b) The minimal central idempotent $z_{\lambda}$ in $A$ indexed by $\lambda \in \hat{A}$ is given by
\[ z_{\lambda} = \sum_{b \in B} t_{\lambda} \chi^\lambda(b^*) b. \]

(c) (Orthogonality of characters) For all $\lambda, \mu \in \hat{A}$,
\[ \sum_{b \in B} \chi^\lambda(b^*) \chi^\mu(b) = \delta_{\lambda\mu} \frac{d_{\lambda}}{t_{\lambda}}. \]
Proof. (a) Since \( \vec{t} \) is nondegenerate, the equation 
\[
\vec{t}(e_{\lambda ij}) = \sum_{\mu \in \hat{A}} t_{\mu} \chi_{\mu}(e_{\lambda ij}) = t_\lambda \delta_{ij}
\]
implies that 
\[
\left\{ e_{\lambda ij} \right\} \text{ is the dual basis to } \left\{ e_{\lambda ji} \right\}
\]
with respect to \( \langle , \rangle \).

Thus, by (???), 
\[
A^\lambda_{ij}(a) = \frac{1}{t_\lambda} \langle a, e_{\lambda ji} \rangle, \quad \text{and so} \quad e_{\lambda ij} = \sum_{b \in B} \langle e_{\lambda ij}, b^* \rangle b = \sum_{b \in B} t_\lambda A^\lambda_{ji}(b^*)b.
\]
(b) By part (a), 
\[
z_\lambda = \sum_{i=1}^{d_\lambda} e_{ii}^\lambda = \sum_{b \in B} t_\lambda \text{Tr}(A^\lambda(b^*))b.
\]
(c) By part (b), 
\[
d_\lambda \delta_{\lambda \mu} = \chi^\mu(z_\lambda) = \sum_{b \in B} t_\lambda \chi^\lambda(b) \chi^\mu(b).
\]

Example 1. Let \( A = \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}) \).

(1) As a left \( A \)-module under the action of \( A \) by left multiplication
\[
A \cong \bigoplus_{\lambda \in \hat{A}} (A^\lambda)^{\oplus d_\lambda},
\]
where \( A^\lambda \) is the irreducible \( A \)-module of column vectors of length \( d_\lambda \).

(2) As an \((A,A)\) bimodule under the action of \( A \) by left and right multiplication
\[
A \cong \bigoplus_{\lambda \in \hat{A}} A^\lambda \otimes \overline{A}^\lambda,
\]
where \( A^\lambda \) is the left \( A \)-module of column vectors of length \( d_\lambda \) and \( \overline{A}^\lambda \) is the right \( A \)-module of row vectors of length \( d_\lambda \).

(3) Let \( a, b \in A \). If \( a \) acts on \( A \) by left multiplication and \( b \) acts on \( A \) by right multiplication then
\[
\text{Tr}(a \otimes b) = \sum_{\lambda \in \hat{A}} \chi^\lambda(a) \chi^\lambda(b),
\]
where \( \chi^\lambda, \lambda \in \hat{A} \), are the irreducible characters of \( A \).

Example 2. Let \( G \) be a finite group and let \( \mathbb{C}G \) be the group algebra of \( G \). The trace of the regular representation of \( \mathbb{C}G \) is given by
\[
\text{tr}(g) = \sum_{h \in G} gh|_h = \begin{cases} |G|, & \text{if } g = 1, \\ 0, & \text{otherwise}. \end{cases}
\]
So, (provided \(|G| \neq 0 \) in \( \mathbb{C} \)) the basis
\[
\left\{ \frac{g^{-1}}{|G|} \right\}_{g \in G}
\]
is the dual basis to \( G \).
with respect to the form $\langle \cdot, \cdot \rangle$ defined by $\text{tr}$. Since $\text{tr}$ is nondegenerate

$$\mathbb{C}G \cong \bigoplus_{\lambda \in \hat{G}} M_{d_\lambda}(\mathbb{C}),$$

for some set $\hat{G}$ and positive integers $d_\lambda$. Then

$$\text{tr} = \sum_{\lambda \in \hat{G}} d_\lambda \chi^\lambda,$$

where $\chi^\lambda$, $\lambda \in \hat{G}$, are the irreducible characters of $G$ and, by (??),

$$z_\lambda = \frac{1}{|G|} \sum_{g \in G} d_\lambda \chi^\lambda(g^{-1})g, \quad \lambda \in \hat{G},$$

are the minimal central idempotents in $\mathbb{C}G$. The orthogonality relation for characters of $G$ (??) is

$$\frac{1}{|G|} \sum_{g \in G} \chi^\lambda(g^{-1})\chi^\mu(g) = \delta_{\lambda\mu}, \quad \text{for } \lambda, \mu \in \hat{G}.$$ 

If $G^\lambda: \mathbb{C}^G \to M_{d_\lambda}(\mathbb{C})$ are the irreducible representations of $G$ then

$$e_{ij}^\lambda = \frac{1}{|G|} \sum_{g \in G} d_\lambda G^\lambda(g^{-1})_{ji}g, \quad \lambda \in \hat{G}, 1 \leq i, j \leq d_\lambda,$$

are a set of matrix units in $\mathbb{C}G$, i.e.

$$e_{ij}^\lambda e_{k\ell}^\mu = \delta_{\lambda\mu} \delta_{kj} e_{\ell i}^\lambda$$

and $\{e_{ij}^\lambda \mid \lambda \in \hat{G}, 1 \leq i, j \leq d_\lambda\}$ is a basis of $\mathbb{C}G$.

Let $g, h \in G$ and let $g$ act on $\mathbb{C}G$ by left multiplication and let $h$ act on $\mathbb{C}G$ by right multiplication. Then

$$\text{Tr}(g \otimes h) = \sum_{k \in G} gkh|_k = \sum_{k \in G} khk^{-1}|_{g^{-1}} = \begin{cases} \text{Card}(C_h), & \text{if } h \text{ is conjugate to } g^{-1}, \\ 0, & \text{otherwise}, \end{cases}$$

where $C_h$ is the conjugacy class of $h$. Thus, by (??),

$$\sum_{\lambda \in \hat{G}} \chi^\lambda(g)\chi^\lambda(h) = \begin{cases} \text{Card}(C_h), & \text{if } h \text{ is conjugate to } g^{-1}, \\ 0, & \text{otherwise}, \end{cases}$$

which is the second orthogonality relation for characters of $G$.

The elements

$$e_g = \sum_{x \in C_g} x$$

are a basis of the center of $\mathbb{C}G$. Since $\{z_\lambda \mid \lambda \in \hat{G}\}$ is also a basis of $Z(\mathbb{C}G)$ we have that

$$\text{Card}(\hat{G}) = \# \text{ of conjugacy classes of } G.$$
though there is no (known) natural bijection between the irreducible representations of $G$ and the conjugacy classes of $G$.

It follows from $\text{???}$ that
\[ |G| = \sum_{\lambda \in \hat{G}} d_{\lambda}^2. \]

Every trace $\bar{t}$ on $\mathbb{C}G$ has a unique decomposition
\[ \bar{t} = \sum_{\lambda \in \hat{G}} t_{\lambda} \chi_{\lambda}, \quad t_{\lambda} \in \mathbb{C}. \]

So, since every $G$-module is semisimple, its decomposition is determined by its character. So

"Two $G$-modules are isomorphic if and only if they have the same character."

and
\[ \dim(Z(\mathbb{C}G)) = (\# \text{ of irreducible representations of } G) = (\# \text{ of conjugacy classes of } G). \]

4. Centralizers.

Let $A$ be an algebra and let $M$ be an $A$-module. The \textit{centralizer} or \textit{commutant} of $M$ is the algebra
\[ \text{End}_A(M) = \{ T \in \text{End}(M) \mid Ta = aT \text{ for all } a \in A \}. \]

If $M$ and $N$ are $A$-modules then $\text{Hom}_A(M, N)$ is a left $\text{End}_A(M)$-module and a right $\text{End}_A(N)$-module.

**Theorem 4.1. (Schur’s Lemma)** Let $A$ be an algebra.

(1) Let $A^\lambda$ be a simple $A$-module. Then $\text{End}_A(A^\lambda) = \mathbb{C} \cdot \text{Id}_{A^\lambda}$.

(2) If $A^\lambda$ and $A^\mu$ are nonisomorphic simple $A$-modules then $\text{Hom}_A(A^\lambda, A^\mu) = \{0\}$.

\textit{Proof}. Let $T: A^\lambda \to A^\mu$ be a nonzero $A$-module homomorphism. Since $A^\lambda$ is simple, $\text{ker} \ T = 0$ and so $T$ is injective. Since $A^\mu$ is simple, $\text{im} \ T = A^\mu$ and so $T$ is surjective. So $T$ is an isomorphism. Thus we may assume that $T: A^\lambda \to A^\lambda$.

When $A^\lambda$ is finite dimensional: Since $\mathbb{C}$ is algebraically closed $T$ has an eigenvector and a corresponding eigenvalue $\alpha \in \mathbb{C}$. Then $T - \alpha \cdot \text{Id} \in \text{Hom}_A(A^\lambda, A^\lambda)$ and so $T - \alpha \cdot \text{Id}$ is either 0 an isomorphism. However, since $\det(T - \alpha \cdot \text{Id}) = 0$, $T - \alpha \cdot \text{Id}$ is not invertible. So $T - \alpha \cdot \text{Id} = 0$. So $T = \alpha \cdot \text{Id}$. So $\text{End}_A(A^\lambda) = \mathbb{C} \cdot \text{Id}$.

When $A^\lambda$ is countable dimensional: We shall show that there exists a $\lambda \in \mathbb{C}$ such that $T - \lambda \cdot \text{Id}$ is not invertible. Suppose $T - \lambda \cdot \text{Id}$ is invertible for all $\lambda \in \mathbb{C}$. Then $p(T)$ is invertible for all polynomials $p(t) \in \mathbb{C}[t]$. So $p(T)/q(T)$ is well defined for all $p(t), q(t) \in \mathbb{C}[t]$.

Let $v \in A^\lambda$ be nonzero. Then the map
\[ \mathbb{C}(t) \longrightarrow \text{End}(V) \longrightarrow V \]
\[ \frac{p(t)}{q(t)} \longrightarrow \frac{p(T)}{q(T)} \longrightarrow \frac{p(T)}{q(T)} v \]
is injective. Since \( \dim \mathbb{C}(t) \) is uncountable and \( \dim V \) is countable this is a contradiction. So \( T - \lambda \cdot \text{Id} \) is invertible for some \( \lambda \in \mathbb{C} \). Then the same proof as in the finite dimensional case shows that \( T = \lambda \cdot \text{Id} \).

If \( A^\lambda \) is unitary: Let
\[
A = \frac{T + T^*}{2} \quad \text{and} \quad B = \frac{T - T^*}{2i}
\]
where \( T^* \) is defined by \( \langle Tv_1, v_2 \rangle = \langle v_1, T^*v_2 \rangle \) for all \( v_1, v_2 \in A^\lambda \). Then
\[
A = A^*, \quad B = B^*, \quad T = A + iB, \quad \text{and} \quad A, B, T \in \text{Hom}_{A^\lambda}(A^\lambda, A^\lambda).
\]
Then the spectral theorem for self adjoint operators says that \( A \) and \( B \) can be diagonalized [Rudin, Thm. 12.2],
\[
A = \sum_i \lambda_i P_i \quad \text{and} \quad B = \sum_j \mu_j Q_j, \quad \text{with} \quad P_i^2 = P_i, \quad Q_j^2 = Q_j, \quad P_i, Q_j \in \text{Hom}_{A^\lambda}(A^\lambda, A^\lambda), \quad \lambda_i, \mu_j \in \mathbb{C}.
\]
Then \( P_i A^\lambda \) is a submodule of \( A^\lambda \). So \( P_i A^\lambda = A^\lambda \). So \( A = \lambda \cdot \text{Id} \).

**Lemma 4.2.** Suppose that \( V \) is a unitary representation. Then
\[
\text{Hom}_{A}(V, V) = \mathbb{C} \cdot \text{Id}_V \quad \text{implies that} \quad V \text{ is irreducible.}
\]

**Proof.** Suppose that \( V \) is not irreducible. Then let \( W \subseteq V \) be a submodule of \( V \). Let
\[
W^\perp = \{ v \in V \mid \langle v, w \rangle = 0, \text{ for all } w \in W \}.
\]
Then \( W^\perp \) is a submodule since, if \( v \in W^\perp \) and \( w \in W \), then \( \langle av, w \rangle = \langle v, a^*w \rangle = 0 \) because \( a^*w \in W \). Now, for Hilbert spaces, we have \( V = W \oplus W^\perp \) and we can define a
\[
\begin{align*}
V & \xrightarrow{P} V \\
w & \mapsto w, \quad \text{if } w \in W, \\
w^\perp & \mapsto 0, \quad \text{if } w \in W^\perp,
\end{align*}
\]
This map is a nonidentity \( A \)-module homomorphism. So \( \text{Hom}_{A}(V, V) \neq \mathbb{C} \cdot \text{Id} \).

**Theorem 4.3.** Let \( A \) be an algebra. Let \( M \) be a semisimple \( A \)-module and set \( Z = \text{End}_{A}(M) \). Suppose that
\[
M \cong \bigoplus_{\lambda \in \hat{M}} (A^\lambda)^{\oplus m_{\lambda}},
\]
where \( \hat{M} \) is an index set for the irreducible \( A \)-modules \( A^\lambda \) which appear in \( M \) and the \( m_{\lambda} \) are positive integers.

(a) \( Z \cong \bigoplus_{\lambda \in \hat{M}} M_{m_{\lambda}}(\mathbb{C}) \).

(b) As an \( (A \otimes Z) \)-module
\[
M \cong \bigoplus_{\lambda \in \hat{M}} A^\lambda \otimes Z^\lambda,
\]
where the $Z^\lambda, \lambda \in \hat{M}$, are the simple $Z$-modules.

**Proof.** Index the components in the decomposition of $M$ by dummy variables $\epsilon_i^\lambda$ so that we may write

$$M \cong \bigoplus_{\lambda \in \hat{M}} \bigoplus_{i=1}^{m_\lambda} A^\lambda \otimes \epsilon_i^\lambda.$$  

For each $\lambda \in \hat{M}, 1 \leq i, j \leq m_\lambda$ let $\phi^\lambda_{ij}: A^\lambda \otimes \epsilon_j \to A^\lambda \otimes \epsilon_i$ be the $A$-module isomorphism given by

$$\phi^\lambda_{ij}(m \otimes \epsilon_j^\lambda) = m \otimes \epsilon_i^\lambda,$$  

for $m \in A^\lambda$.

By Schur’s Lemma,

$$\text{End}_A(M) = \text{Hom}_A(M, M) \cong \text{Hom}_A \left( \bigoplus_{\lambda \in \hat{M}} \bigoplus_{j} A^\lambda \otimes \epsilon_j^\lambda, \bigoplus_{\mu} \bigoplus_{i} A^\mu \otimes \epsilon_i^\mu \right) \cong \bigoplus_{\lambda, \mu} \bigoplus_{i,j} \delta_{\lambda\mu} \text{Hom}_A(A^\lambda \otimes \epsilon_j^\lambda, A^\mu \otimes \epsilon_i^\mu) \cong \bigoplus_{\lambda} \bigoplus_{i,j=1}^{m_\lambda} \mathbb{C} \phi^\lambda_{ij}.$$

Thus each element $z \in \text{End}_A(M)$ can be written as

$$z = \sum_{\lambda \in \hat{M}} \sum_{i,j=1}^{m_\lambda} z^\lambda_{ij} \phi^\lambda_{ij}, \quad \text{for some } z^\lambda_{ij} \in \mathbb{C},$$

and identified with an element of $\bigoplus_{\lambda} M_{m_\lambda}(\mathbb{C})$. Since $\phi^\lambda_{ij} \phi^\mu_{kl} = \delta_{\lambda\mu} \delta_{jk} \phi^\lambda_{il}$ it follows that

$$\text{End}_A(M) \cong \bigoplus_{\lambda \in \hat{M}} M_{m_\lambda}(\mathbb{C}).$$

(b) As a vector space $Z^\mu = \text{span}\{\epsilon_i^\mu \mid 1 \leq i \leq m_\mu\}$ is isomorphic to the simple $\bigoplus_{\lambda} M_{m_\lambda}(\mathbb{C})$ module of column vectors of length $m_\mu$. The decomposition of $M$ as $A \otimes Z$ modules follows since

$$(a \otimes \phi^\lambda_{ij})(m \otimes \epsilon_k^\mu) = \delta_{\lambda\mu} \delta_{jk} (a \otimes \epsilon_k^\mu), \quad \text{for all } m \in A^\mu, a \in A,$$

If $A$ is an algebra then $A^{op}$ is the algebra $A$ except with the opposite multiplication, i.e.

$$A^{op} = \{a^{op} \mid a \in A\} \quad \text{with} \quad a_1^{op} a_2^{op} = (a_2 a_1)^{op}, \quad \text{for all } a_1, a_2 \in A.$$  

Let left regular representation of $A$ is the vector space $A$ with $A$ action given by left multiplication. Here $A$ is serving both as an algebra and as an $A$-module. It is often useful to distinguish the two roles of $A$ and use the notation $\bar{A}$ for the $A$-module, i.e. $\bar{A}$ is the vector space

$$\bar{A} = \{\bar{b} \mid b \in A\} \quad \text{with } A\text{-action} \quad \bar{a} \bar{b} = \bar{ab}, \quad \text{for all } a \in A, \bar{b} \in \bar{A}.$$
Proposition 4.4. Let $A$ be an algebra and let $\tilde{A}$ be the regular representation of $A$. Then $\text{End}_A(\tilde{A}) \cong A^{\text{op}}$. More precisely,

$$\text{End}_A(\tilde{A}) = \{ \phi_b \mid b \in A \},$$

where $\phi_b$ is given by $\phi_b(\tilde{a}) = \tilde{ab}$, for all $\tilde{a} \in \tilde{A}$.

**Proof.** Let $\phi \in \text{End}_A(\tilde{A})$ and let $b \in A$ be such that $\phi(\tilde{1}) = \tilde{b}$. For all $\tilde{a} \in \tilde{A}$,

$$\phi(\tilde{a}) = \phi(a \cdot \tilde{1}) = a\phi(\tilde{1}) = a\tilde{b} = \tilde{ab},$$

and so $\phi = \phi_b$. Then $\text{End}_A(\tilde{A}) \cong A^{\text{op}}$ since

$$(\phi_{b_1} \circ \phi_{b_2})(\tilde{a}) = a\tilde{b_2}b_1 = \phi_{b_2b_1}(\tilde{a}),$$

for all $b_1, b_2 \in A$ and $\tilde{a} \in \tilde{A}$. $\blacksquare$

5. Characterizing algebras isomorphic to $\bigoplus_\lambda M_{d_\lambda}(\mathbb{C})$

**Theorem 5.1.** Suppose that $A$ is an algebra such that the regular representation $\tilde{A}$ of $A$ is completely decomposable. Then $A$ is isomorphic to a direct sum of matrix algebras, i.e.

$$A \cong \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}),$$

for some set $\hat{A}$ and some positive integers $d_\lambda$, indexed by the elements of $\hat{A}$.

**Proof.** If $\tilde{A}$ is completely decomposable then, by Theorem ???, $\text{End}_A(\tilde{A})$ is isomorphic to a direct sum of matrix algebras. By Proposition ??,

$$A^{\text{op}} \cong \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C}),$$

for some set $\hat{A}$ and some positive integers $d_\lambda$, indexed by the elements of $\hat{A}$. The map

$$\bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})^{\text{op}} \xrightarrow{a^t} \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$$

where $a^t$ is the transpose of the matrix $a$, is an algebra isomorphism. So $A$ is isomorphic to a direct sum of matrix algebras. $\blacksquare$

**Proposition 5.2.** Let $A = \bigoplus_{\lambda \in \hat{A}} M_{d_\lambda}(\mathbb{C})$. Then the trace $\text{tr}$ of the regular representation of $A$ is nondegenerate.

**Proof.** As $A$-modules, the regular representation

$$\tilde{A} \cong \bigoplus_{\lambda \in \hat{A}} (A^{\lambda})^{\otimes d_\lambda},$$

where $A^\lambda$ is the irreducible $A$-module consisting of column vectors of length $d_\lambda$. So the trace $tr$ of the regular representation is given by

$$tr = \sum_{\lambda \in \hat{A}} d_\lambda \chi^\lambda,$$

where $\chi^\lambda$ are the irreducible characters of $A$. Since the $d_\lambda$ are all nonzero the trace $tr$ is nondegenerate.

**Theorem 5.3.** (Maschke’s theorem) Let $A$ be an algebra such that the trace $tr$ of the regular representation of $A$ is nondegenerate. Then every representation of $A$ is completely decomposable.

**Proof.** Let $B$ be a basis of $A$ and let $B^*$ be the dual basis of $A$ with respect to the form $\langle,\rangle: A \times A \to \mathbb{C}$ defined by

$$\langle a_1, a_2 \rangle = tr(a_1 a_2), \quad \text{for all } a_1, a_2 \in A.$$

The dual basis $B^*$ exists because the trace $tr$ is nondegenerate.

Let $M$ be an $A$-module. If $M$ is irreducible then the result is vacuously true, so we may assume that $M$ has a proper submodule $N$. Let $p \in \text{End}(M)$ be a projection onto $N$, i.e. $pM = N$ and $p^2 = p$. Let

$$[p] = \sum_{b \in B} bp^b, \quad \text{and} \quad e = \sum_{b \in B} bb^*.$$

For all $a \in A$,

$$tr(ea) = \sum_{b \in B} tr(bb^*a) = \sum_{b \in B} \langle ab, b^* \rangle = \sum_{b \in B} ab \big|_b = tr(a),$$

So $tr((e - 1)a) = 0$, for all $a \in A$. Thus, since $tr$ is nondegenerate, $e = 1$.

Let $m \in M$. Then $pb^*m \in N$ for all $b \in B$, and so $[p]m \in N$. So $[p]M \subseteq N$. Let $n \in N$. Then $pb^*n = b^*n$ for all $b \in B$, and so $[p]n = en = 1 \cdot n = n$. So $[p]M = N$ and $[p]^2 = [p]$, as elements of $\text{End}(M)$.


$$M = [p]M \oplus (1 - [p])M = N \oplus [1 - p]M,$$

and, by Proposition ??, $[1 - p]M$ is an $A$-module. So $[1 - p]M$ is an $A$-submodule of $M$ which is complementary to $M$. By induction on the dimension of $M$, $N$ and $[1 - p]M$ are completely decomposable, and therefore $M$ is completely decomposable. $lacksquare$

Together, Theorems ??, ?? and Proposition ?? yield the following theorem.

**Theorem 5.4.** (Artin-Wedderburn) Let $A$ be a finite dimensional algebra over $\mathbb{C}$. The following are equivalent:

1. Every representation of $A$ is completely decomposable.
2. The trace of the regular representation of $A$ is nondegenerate.
3. The regular representation of $A$ is completely decomposable.

**Example 1.** Let $A$ be the algebra with basis $\{1, e\}$ and multiplication given by $e^2 = 0$. Then

$$\tilde{t}: A \to \mathbb{C} \quad \text{given by} \quad \tilde{t}(a + be) = a + b$$
is a nondegenerate trace on $A$. The regular representation of $A$ is given by

$$\tilde{A}(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{A}(e) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $\mathbb{C}e$ is the only submodule of $\tilde{A}$. Thus, $\tilde{A}$ is not completely decomposable. The trace $\text{tr}$ of the regular representation of $A$ is given by

$$\text{tr}(a + be) = 2a, \quad \text{for } a, b \in \mathbb{C}.$$

**Theorem 5.5.** *(Burnside’s Theorem)* Let $A$ be an algebra and let $M : A \to \text{End}(M)$ be an irreducible representation of $A$. Then $M(A) = \text{End}(M)$.

**Proof.** Clearly, $M(A) \subseteq \text{End}(M)$ and $M$ is both a simple $M(A)$-module and a simple $\text{End}(M)$-module. As $\text{End}(M)$-modules

$$\text{End}(M) \cong M^\oplus d,$$

and so, by restriction, this is also true as an $M(A)$-module. Thus, by Schur’s lemma,

$$\text{End}_{M(A)}(\text{End}(M)) = M_d(\mathbb{C}).$$

Let us label the summands in the decomposition by dummy variables $\epsilon_i$,

$$\frac{\text{End}(M)}{M} = \bigoplus_{i=1}^{d} M \otimes \epsilon_i,$$

so that $E_{ii}(\text{End}(M)) = M \otimes \epsilon_i$.

Now $\overline{M(A)} \subseteq \text{End}(M)$ is an $M(A)$ submodule of $\text{End}(M)$. However,

$$E_{ii}(\text{End}(M)) \subseteq M \otimes \epsilon_i \quad \text{and} \quad \overline{M(A)} = E_{11}M(A) \oplus \cdots \oplus E_{dd}M(A) \subseteq M \otimes \epsilon_1 \oplus \cdots \oplus M \otimes \epsilon_d.$$

Since $M$ is a simple $M(A)$ module, each $E_{ii}\overline{M(A)}$ is isomorphic to $M$ or 0. So

$$\overline{M(A)} \cong M^\oplus k, \quad \text{for some } 1 \leq k \leq d.$$

So the regular representation of $M(A)$ is semisimple and $M(A) \cong M_k(\mathbb{C})$. Since $\text{dim}(M) = d$ and $M$ is a simple module for $M(A)$ we have $M(A) \cong M_d(\mathbb{C})$. So $M(A) = \text{End}(M)$. \(\blacksquare\)

**Remark 1.** We used Schur’s lemma in a crucial way so we are assuming that $\mathbb{C}$ is algebraically closed. In general we can say:

If $M$ is a simple $A$-module then $M(A) = \text{End}_Z(M)$ where $Z = \text{End}_A(M)$.

The proof is similar to that given above and is called the Jacobson density theorem.

**Example.** Assume that $A$ is a commutative algebra and let $M$ be a simple $A$-module. Then $M(A)$ is commutative and $M(A) = \text{End}(M) \cong M_d(\mathbb{C})$, where $d = \text{dim}(M)$. However, $M_d(\mathbb{C})$ is commutative if and only if $d = 1$. This shows that every irreducible representation of a commutative algebra is one dimensional.

**Example 2.** Explain what the error is in the following proof of Burnside’s theorem: If $M$ is an irreducible $A$-module then $M(A) = \text{End}(M)$.

**Proof.** Let $\{m_1, \ldots, m_d\}$ be a basis of $M$. Since $M$ is irreducible, for any $i$ and $j$ there is an $a \in A$ such that $M(a)m_j = m_i$. So the matrix $E_{ji} \in M(A)$ for all $1 \leq i, j \leq n$. So $\text{End}(M) \subseteq M(A)$. So $M(A) = \text{End}(M)$. \(\blacksquare\)