Calculus is the study of

1. Derivatives
2. Integrals
3. Applications of derivatives
4. Applications of integrals

A derivative is a creature you put a function into, it chews on it, and spits out a new function. A function takes in a number, chews on it, and spits out a new number.

\[
\begin{array}{cc}
\text{Derivatives} & \text{Functions} \\
\text{input function} & \longrightarrow & \frac{d}{dx} & \longrightarrow & \text{output function} \\
\text{input number} & \longrightarrow & f & \longrightarrow & \text{output number}
\end{array}
\]

The integral is the derivative backwards:

Numbers are at the bottom of the food chain.

At some point humankind wanted to count things and discovered the positive integers,

\[1, 2, 3, 4, 5, \ldots\]

GREAT for counting something,

BUT what if you don’t have anything? How do we talk about nothing, nulla, zilch?

… and so we discovered the nonnegative integers,

\[0, 1, 2, 3, 4, 5, \ldots\]

GREAT for adding,

\[5 + 3 = 8, \ 0 + 10 = 10, \ 21 + 37 = 48,\]

BUT not so great for subtraction,

\[5 - 3 = 2, \ 2 - 0 = 2, \ 12 - 34 =???.\]
... and so we discovered the **integers**

...$\ldots$,$-3,-2,-1,0,1,2,3,\ldots$.

GREAT for adding, subtracting and multiplying,

\[3 \cdot 6 = 18, \quad -3 \cdot 2 = -6, \quad 0 \cdot 7 = 0,\]

BUT not so great if you only want part of the sausage $\ldots$,

... and so we discovered the **rational numbers**, $\frac{a}{b}$, $a$ an integer, $b$ an integer, $b \neq 0$.

GREAT for addition, subtraction, multiplication, and division,

BUT not so great for finding $\sqrt{2} = \ldots$,

... and so we discovered the **real numbers**, all decimal expansions.

Examples:

\[\pi = 3.1415926\ldots,\]
\[e = 2.71828\ldots,\]
\[\sqrt{2} = 1.414\ldots,\]
\[10 = 10.0000\ldots,\]

GREAT for addition, subtraction, multiplication, and division,

BUT not so great for finding $\sqrt{-9} = \ldots$,

... and so we discovered the **complex numbers**, $a + bi$, $a$ a real number, $b$ a real number, $i = \sqrt{-1}$.

Examples:

\[3 + \sqrt{2}i, \quad 6 = 6 + 0i, \quad \pi + \sqrt{7}i,\]

and

\[\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9\sqrt{-1}} = 3i.\]

GREAT.

*Addition:* $\quad (3 + 4i) + (7 + 9i) = 3 + 7 + 4i + 9i = 10 + 13i.$

*Subtraction:* $\quad (3 + 4i) - (7 + 9i) = 3 - 7 + 4i - 9i = -4 - 5i.$

*Multiplication:*

\[
(3 + 4i)(7 + 9i) = 3(7 + 9i) + 4i(7 + 9i)
= 21 + 27i + 28i + 36i^2
= 21 + 55i - 36
= -15 + 55i.
\]
\[ \frac{3 + 4i}{7 + 9i} = \frac{(3 + 4i)(7 - 9i)}{(7 + 9i)(7 - 9i)} = \frac{21 - 27i + 28i + 36}{49 - 63i + 63i + 81} = \frac{57 + i}{130} = \frac{57}{130} + \frac{1}{130}i. \]

**Square Roots:** We want \( \sqrt{-3 + 4i} \) to be some \( a + bi \).

If \( \sqrt{-3 + 4i} = a + bi \)

then

\[ -3 + 4i = (a + bi)^2 = a^2 + abi + abi + b^2i^2 = a^2 - b^2 + 2abi. \]

So

\[ a^2 - b^2 = -3 \quad \text{and} \quad 2ab = 4. \]

Solve for \( a \) and \( b \).

\[ b = \frac{4}{2a} = \frac{2}{a}. \]

So \( a^2 - \left(\frac{2}{a}\right)^2 = -3. \)

So \( a^2 - \frac{4}{a^2} = -3. \)

So \( a^4 - 4 = -3a^2. \)

So \( a^4 + 3a^2 - 4 = 0. \)

So \( (a^2 + 4)(a^2 - 1) = 0. \)

So \( a^2 = -4 \) or \( a^2 = 1. \)

So \( a = \pm 1, \) and \( b = \frac{2}{\pm 1} = 2 \) or \(-2. \)

So \( a + bi = 1 + 2i \) or \( a + bi = -1 - 2i. \)

So \( \sqrt{-3 + 4i} = \pm(1 + 2i). \)

**Graphing:**

**Factoring:**

\[ x^2 + 5 = (x + \sqrt{5}i)(x - \sqrt{5}i), \]
\[ x^2 + 1 = \left(x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right)\left(x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right) \]

This is REALLY why we like the complex numbers. The **fundamental theorem of algebra** says that ANY POLYNOMIAL (for example, \( x^{12673} + 2563x^{159} + \pi x^{121} + \sqrt{7}x^{23} + 9621\frac{1}{2} \)) can be factored completely as

\( (x - u_1)(x - u_2) \cdots (x - u_n) \)

where \( u_1, \ldots, u_n \) are complex numbers.