1 Weyl characters

Your favourite group $G^\vee$ (probably $SL_3(\mathbb{C})$) corresponds to

$W = \{ \text{chambers} \}$

and

$P^\vee = \{ \text{dots} \}$

The irreducible $G^\vee$-modules $L(\lambda^\vee)$ are indexed by $\lambda^\vee \in (P^\vee)^+$ and

\[
\text{char}(L(\lambda^\vee)) = \sum_{\mu^\vee \in P^\vee} \text{Card}(B(\lambda^\vee)_{\mu^\vee} x^{\mu^\vee}),
\]

where

$B(\lambda^\vee)_{\mu^\vee} = \{ \text{Littelmann paths of type } \lambda^\vee \text{ and end } \mu^\vee \}$.

If

$G = G(\mathbb{C}(t)), \quad K = G(\mathbb{C}[[t]]), \quad \text{and} \quad U^− = \left\{ \begin{pmatrix} 1 & \cdots & 0 \\ \ast & \ddots & \ast \\ & \cdots & 1 \end{pmatrix} \right\}$.

then $G/K$ is the loop Grassmanian and

$G = \bigsqcup_{\lambda^\vee \in (P^\vee)^+} Kt_{\lambda^\vee} K \quad \text{and} \quad G = \bigsqcup_{\mu^\vee \in P^\vee} U^− {t_{\mu^\vee}} K.$
The MV cycles of type $\lambda^\vee$ and weight $\mu^\vee$ are the elements of

$$MV(\lambda^\vee)_{\mu^\vee} = \{\text{irreducible components of } Kt_{\lambda^\vee}K \cap U^{-t_{\mu^\vee}K}\},$$

and

$$\text{char}(L(\lambda^\vee)) = \sum_{\mu^\vee} \text{Card}(MV(\lambda^\vee)_{\mu^\vee})x^{\mu^\vee}.$$ 

2 Hecke algebras

The spherical and affine Hecke algebras are

$$\tilde{H}_{\text{sph}} = C(K\backslash G/K) \quad \text{and} \quad \tilde{H} = C(I\backslash G/I),$$

where

$$G = G(\mathbb{C}((t))) \quad \text{and} \quad K = G(\mathbb{C}[[t]]) \quad \text{where} \quad B = \left\{ \begin{pmatrix} * & \cdots & * \\ 0 & \cdots & * \end{pmatrix} \right\}.$$

The Satake map is

$$\mathbb{C}[X]^W = Z(\tilde{H}) \sim Z(\tilde{H})1_0 = 1_0\tilde{H}1_0 = \tilde{H}_{\text{sph}}$$

$$P_{\lambda^\vee} \leftarrow 1_0X^{\lambda^\vee}1_0 = \chi_Kt_{\lambda^\vee}K \quad \text{"obvious" basis}$$

and $P_{\lambda^\vee}$ are the Hall-Littlewood polynomials.

$$P_{\lambda^\vee} = \sum_{\mu^\vee \in P_{\lambda^\vee}} \text{Card}_q(P(\lambda^\vee)_{\mu^\vee})x^{\mu^\vee},$$

where

$$P(\lambda^\vee)_{\mu^\vee} = \{\text{Hecke paths of type } \lambda^\vee \text{ and end } \mu^\vee\} \hookrightarrow \{\text{slices of } G/K \text{ in } Kt_{\lambda^\vee}K \cap U^{-t_{\mu^\vee}K}\}$$

and

$$\text{Card}_q(P(\lambda^\vee)_{\mu^\vee}) = \sum_{p \in P(\lambda^\vee)_{\mu^\vee}} (# \text{ of } \mathbb{F}_q \text{ points in slice } p).$$

After normalization,

$$P_{\lambda^\vee}\big|_{q^{-1}=0} = \text{char}(L(\lambda^\vee)).$$

3 Buildings

The group $B$ is a Borel subgroup of $G = G(\mathbb{C})$ and

$$G/B = \text{flag variety} = \text{building}.$$ 

The cell decomposition of $G/B$ is

$$G = \bigsqcup_{w \in W} BwB.$$
Idea: The points of $W$ are regions, or chambers.

$$W = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1s_2s_1 = s_2s_1s_2 \rangle$$

If $w = s_{i_1} \cdots s_{i_t}$ is a minimal length path to $w$ then

$$BwB = \{x_{i_1}(c_1)s_{i_1} \cdots x_{i_t}(c_t)s_{i_t}B \mid c_1, \ldots, c_t \in \mathbb{C}\},$$

where $x_i(c) = 1 + cE_{i,i+1}$, with $E_{i,i+1}$ the matrix with a 1 in the $(i,i+1)$ entry and all other entries 0.

**IDEA:** The points of $G/B$ are regions, or chambers.

Just as the building of $W$, the *Coxeter complex*, has relations

$$s_1s_2s_1 = s_2s_1s_2$$

the building of $G/B$ also has relations

$$x_1(c_1)s_1x_2(c_2)s_2x_1(c_3)s_1 = x_2(c_3)s_2x_1(c_1c_3 - c_2)s_1x_2(c_3)s_2$$

An *apartment* is a subbuilding of $G/B$ that looks like $W$.

The Borel subgroup of $G = G(\mathbb{C}(t))$ is $I$ and

$$G/I$$

is the *affine flag variety*.
with
\[ G = \bigsqcup_{w \in W} IwI, \quad \text{where} \quad \hat{W} = W \rtimes P^\vee \]

is the affine Weyl group

The affine building \( G/I \) has sectors

\[ \text{since } G = \bigsqcup_{v \in \mathbb{W}} U^- vI. \]

4 MV polytopes

Let
\[ T = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\} \]
and let \( V \) be a \( T \)-module

with \( T \)-invariant inner product \( \langle \cdot, \cdot \rangle \) (such that \( \langle v, v \rangle = 0 \) \( \Leftrightarrow v = 0 \)). Let
\[ \mathfrak{h} = \text{Lie}(T) \quad \text{and} \quad \mathbb{P}V = \{[v] \mid v \in V, v \neq 0\}, \]

where \([v] = \text{span}\{v\}\). The moment map on \( \mathbb{P}V \) is
\[ \mu: \mathbb{P}V \to \mathfrak{h}^* \quad \mu_v \quad \text{where} \quad \mu_v(h) = \frac{\langle hv, v \rangle}{\langle v, v \rangle}. \]

Now let \( V = L(\gamma) \) be a simple \( G \)-module \( (G = G(\mathbb{C})) \) with highest weight vector \( v^+ \). Then
\[ B[v^+] = [v^+] \quad \text{and} \quad G[v^+] \subseteq \mathbb{P}V \]

is the image of \( G/B \) in \( \mathbb{P}V \). The moment map on \( G/B \) (associated to \( \gamma \)) is
\[ \mu: G/B \to \mathbb{P}V \to \mathfrak{h}^* \quad gB \to g[v^+] \to \mu_{g v^+}. \]

Joel(Kamnitzer)'s favourite case is \( G/K \) with \( \gamma = \omega_0 \) (the fundamental weight corresponding to the added node on the extended Dynkin diagram) and
\[ \mu(\text{MV cycle of type } \lambda^\vee \text{ and weight } \mu^\vee) = (\text{MV polytope of type } \lambda^\vee \text{ and weight } \mu^\vee) \]
5 Tropicalization

Let $G = G(\mathbb{C}((t)))$.

$\mathbb{C}((t)) = \{ a_\ell t^\ell + a_{\ell+1} t^{\ell+1} + \cdots \mid \ell \in \mathbb{Z}, a_i \in \mathbb{C} \}$.

Points of $G/I$ are $gI$, where $g = (g_{ij})$, $g_{ij} \in \mathbb{C}((t))$.

The valuation on $\mathbb{C}((t))$

$v(a_\ell t^\ell + a_{\ell+1} t^{\ell+1} + \cdots) = \ell$,

is like log

$v(f_1f_2) = v(f_1) + v(f_2)$ and $v(f_1 + f_2) = \min(v(f_1), v(f_2))$.

Then $v(gI)$ is a tropical point of $v(G/I)$, the tropical flag variety. An amoeba, or tropical subvariety, is the image, under $v$, of a subvariety of $G/I$. 

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