Heisenberg group

\[ \mathfrak{h}_2 \] dual \( \mathbb{R} \)-vector spaces, \( \langle , \rangle : \mathfrak{h}_2 \times \mathfrak{h}_2 \to \mathbb{R} \)

The group algebras are

\[ K \mathfrak{h}(pt) = \text{span} \{ x^\mu | \mu \in \mathfrak{h}_2^* \}, \quad K \mathfrak{v}(pt) = \text{span} \{ y^\lambda | \lambda \in \mathfrak{h}_2^* \} \]

with

\[ x^\mu x^\nu = x^{\mu + \nu} \quad \text{and} \quad y^\lambda y^\xi = y^{\lambda + \xi} \]

If \( \xi_1, \ldots, \xi_n \) is a basis of \( \mathfrak{h}_2^* \)

\( \xi_1, \ldots, \xi_n \) a basis of \( \mathfrak{h}_2 \)

then

\[ K \mathfrak{h}(pt) = \mathbb{C}[x_1^{\xi_1}, \ldots, x_n^{\xi_n}] \quad \text{and} \quad K \mathfrak{v}(pt) = \mathbb{C}[y_1^{\xi_1}, \ldots, y_n^{\xi_n}] \]

with

\[ x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}, \quad \text{if} \quad \mu = \mu_1 \xi_1 + \cdots + \mu_n \xi_n \]

\[ y^\lambda = y_1^{\lambda_1} \cdots y_n^{\lambda_n}, \quad \text{if} \quad \lambda = \lambda_1 \xi_1 + \cdots + \lambda_n \xi_n \]

let \( q^{1/n} \) be a parameter.

The Heisenberg group is

\[ \mathbb{H} = \{ q^{1/n} x^\mu y^\lambda | x \in \mathfrak{h}_2, \mu \in \mathfrak{h}_2^*, \lambda \in \mathfrak{h}_2^* \} \]

with (*) and

\[ x^\mu y^\lambda = q^{\langle \mu, \lambda \rangle} y^\lambda x^\mu. \]
Weyl algebras

\[ \mathfrak{g} \]
\[ \mathfrak{g}^* \]
dual vector spaces, \( \langle , \rangle : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{C} \).

The symmetric algebras are

\[ H_+ (\mathfrak{g}^*) = \mathbb{C} [x_1, \ldots, x_n] \]
\[ H_- (\mathfrak{g}^*) = \mathbb{C} [D_1, \ldots, D_n] \]

with

\[ x_{\mu} = x_1^1 + \cdots + x_n^\mu, \quad \text{if} \quad \mu = \mu_1 \xi_1 + \cdots + \mu_n \xi_n \]
\[ D_{\mu} = D_1^\mu + \cdots + D_n^\mu, \quad \text{if} \quad \mu = \lambda_1 \xi_1^\mu + \cdots + \lambda_n \xi_n \]

Let \( \kappa \) be a parameter.

The Weyl algebra \( D \) is generated by

\[ \mathbb{C} [D_1, \ldots, D_n] \] and \( \mathbb{C} [x_1, \ldots, x_n] \)

with

\[ D_{\mu} x_{\mu} = x_{\mu} D_{\mu} + \kappa \langle \mu, \nu \rangle. \]

\( D \) acts on polynomials: If \( \langle \xi_i, \xi_j \rangle = \delta_{ij}, \kappa = 1 \)

\[ D_j = \frac{\partial}{\partial x_j}, \quad \text{then} \quad \left[ \frac{\partial}{\partial x_j}, x_i \right] = \frac{\partial}{\partial x_j} x_i - x_i \frac{\partial}{\partial x_j} = \delta_{ij}. \]

In physics, sometimes \( \kappa = i \hbar \).
Rational Cherednik algebras

$W_0$ is a finite subgroup of $GL(V_e)$ generated by

$R^+ = \{ s \in W_0 \mid s \text{ is a reflection} \}$

The group algebra is

$CW_0 = \text{span} \{ tw \mid w \in W_0 \}$ with $tw \cdot tw' = tww'.$

($W_0$ acts on $V_e$ by $\langle w \mu, \lambda \nu \rangle = \langle \mu, w^{-1} \lambda \nu \rangle$.)

For $s \in R^+$ fix $x_s \in Z^*_e$ and $x_s \in Z_e$ so that

$s_{\mu} = \mu - \langle \mu, x_s \rangle x_s$ and $s^{-1}_{\lambda \nu} = \lambda \nu - \langle \lambda \nu, x_s \rangle x_s,$

$wsw^{-1} = ws$ and $x_{wsw^{-1}} = x_s$ for $w \in W_0.$

Fix parameters

$c_s, s \in R^+$ with $c_s = c_dwsw^{-1}$ for $w \in W_0.$

The rational Cherednik algebra $H$ is gen. by

$D, D_1, \ldots, D_n, x_1, \ldots, x_n$ and $CW_0$

with

$tw x_{\mu} = x_{w\mu} tw, \quad tw D_{\lambda \nu} = D_{w\lambda \nu} tw$

$D_{\lambda \nu} x_{\mu} = x_{\mu} D_{\lambda \nu} + K \langle \mu, \lambda \nu \rangle - \sum_{s \in R^+} c_s \langle \lambda \nu, x_s \rangle \langle x_s, x_{\mu} \rangle x_s.$
Dunkl operators

For $p \in \mathbb{C}[x_1, \ldots, x_n]$, 

$$D_{\lambda} p = pD_{\lambda} - \sum_{s \in \mathbb{R}^+} c_s \langle \lambda, s \rangle (\Delta_s p) t_s$$

where $d_{\lambda} : \mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ is given by 

$$d_{\lambda}(e_\mu) = \langle \mu, \lambda \rangle, \quad d_{\lambda}(p_\mu) = p_\mu d_{\lambda}(p_\mu) + d_{\lambda}(p_\mu)p_\mu,$$

and $\Delta_s : H \to H$ is 

$$\Delta_s = \mathbb{R}^+ P_s, \, \text{the BGG-operator}.$$

The subalgebra 

$H$ generated by $\mathbb{C}e_0$ and $\mathbb{C}[D, \ldots, D_n]$ 

has a 1-dim module spanned by given by 

$$t_w H = H \quad \text{and} \quad D_{\lambda} t_w H = 0.$$

The polynomial representation of $\hat{H}$ is 

$$\text{Ind}_{\mathbb{R}^+}^H(\hat{H}) = \mathbb{C}\hat{H} = \mathbb{C}[x_1, \ldots, x_n] \hat{H}.$$

$D_{\lambda}$ acts on $\hat{H}$ by the Dunkl operator 

$$D_{\lambda} = k d_{\lambda} - \sum_{s \in \mathbb{R}^+} c_s \langle \lambda, s \rangle \frac{1}{t_s} (1-s).$$
The trigonometric Cherednik algebra $\tilde{H}_n$

$W_0$ is a finite subgroup of $GL(V)$

generated by $R^+$. Then $W_0$ has a presentation by

generators $s_1, \ldots, s_n$ and relations

$$s_i^2 = 1 \quad \text{and} \quad s_i s_j s_i \cdots = s_j s_i s_j \cdots$$

where $s_i s_j = \frac{s_i s_j}{s_j s_i}$.

Let $CE[y_1, \ldots, y_n] = S(V)$ with

$$y_1 \cdot y_\nu = \lambda_1 y_\nu + \cdots + \lambda_n y_\nu$$

if $\lambda_1 y_\nu + \cdots + \lambda_n y_\nu$.

The trigonometric Hecke algebra $H_n$ is gen. by

$CE[y_1, \ldots, y_n]$, $CE[x_1^\pm 1, \ldots, x_n^\pm 1]$ and $CE W_0$

with

$$t_1 X_1 = X_1 t_1 X_1,$$

$$t_i y_\nu = y_\nu t_i + C_i \langle Y,\nu \rangle,$$

for $i = 1, \ldots, n$

$$y_\nu X_1 = X_1 y_\nu + \lambda_1 y_\nu X_1 - \sum_{s \in R^+} C_s \langle \lambda_\nu, s_\nu \rangle \frac{X_1 X_{s_\nu}}{1 - X_1 s_\nu}.$$
Dunkl-Cherednik operators

Note: $K_r(pt) = \mathbb{C}[X_1, \ldots, X_n]$ and $\pi_5: K_r(pt) \to K_r(pt)$,

\[
\pi_5 X^\mu = \frac{X^\mu - X_0^\mu}{1 - X_0}
\]

is a Demazure operator.

The subalgebra $H_{pr}$

$H_{pr}$ generated by $C\mathcal{W}_0$ and $C[\mathcal{Y}_i, \ldots, \mathcal{Y}_n]$ has a 1-dim'l module spanned given by

\[
t_{\nu}\emptyset = \emptyset \quad \text{and} \quad Y_{\lambda_{\nu}} \emptyset = \langle \nu, \emptyset \rangle \emptyset
\]

where $\langle \nu, \emptyset \rangle = \mathcal{C}_i$ for $i = 1, \ldots, n$.

The polynomial representation of $H_{pr}$ is

\[
\text{Ind}_{H_{pr}}^{\mathcal{H}_r}(\emptyset) = H_{pr}\emptyset = C[\mathcal{X}_1^{\pm 1}, \ldots, X_n^{\pm 1}, \emptyset].
\]

The Dunkl-Cherednik operator is

\[
Y_{\lambda_{\nu}} = \langle \nu, \emptyset \rangle + \kappa \delta_{\lambda_{\nu}} - \sum_{s \in \mathcal{R}^+} \mathcal{C}_s \langle \lambda_{\nu}, \emptyset \rangle \frac{1}{1 - X_0^s}(1 - s)
\]

where $\delta_{\lambda_{\nu}}: \mathbb{C}[\mathcal{X}_1^{\pm 1}, \ldots, X_n^{\pm 1}] \to \mathbb{C}[\mathcal{X}_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ is given by

\[
\delta_{\lambda_{\nu}}(X^\mu) = \langle \mu, \emptyset \rangle X^\mu.
\]
Quantisation

\[ H_\nu (pt) = C[\nu_1, \ldots, \nu_n] \xrightarrow{\text{ch}} X_\nu (pt) = C[\nu_1^*, \ldots, \nu_n^*] \]

\[ e^{X_\mu} \xleftarrow{} X_\mu \]

let

\[ X_\mu = e^{h \nu_\mu} \]

Then \( \nu_\mu = \frac{1}{h} d_\mu \)

and

\[ \text{ch} \rightarrow \tilde{H}_\nu^{\tilde{\mu}} \]

\[ e^{h \nu_\mu} \xleftarrow{} X_\mu \]

\[ \frac{1}{h} d_\mu \xleftarrow{} \nu_\mu \]

\[ t_\nu \xleftarrow{} t_\nu \]

is an "isomorphism" with

\[ S_\nu = \left< \rho_\nu, X_\nu \right> + \frac{1}{h} D_\nu + \sum_{s \in \mathbb{R}^+} \left< \nu_\mu, \nu_\nu \right> \frac{1}{h \nu} \left( 1 - \frac{h \xi s}{1 - e^{h \xi s}} \right) (1 - t_\nu) \]