Symmetric functions

\[ \mathfrak{h}_d \] dual \( \mathbb{Z} \)-vector spaces, \( \langle \mathfrak{h}_d, \mathfrak{h}_d^* \rangle \rightarrow \frac{1}{2} \mathbb{Z} \)

\( W_0 \) a finite subgroup of \( S_\mathfrak{h}(\mathfrak{h}_d) \) generated by reflections. Then \( W_0 \) acts on the group algebra

\[ K_{\mathfrak{h}_d}(pt) = \text{span} \{ y^{\lambda} \mid \lambda \in \mathfrak{h}_d \} \]

with \( y^{\lambda} y^{\mu} = y^{\lambda + \mu} \) by \( w y^{\lambda} = y^{w \lambda} \).

The algebra of symmetric functions is

\[ K_{\mathfrak{h}_d}(pt)^{W_0} = \{ f \in K_{\mathfrak{h}_d}(pt) \mid wf = f, \text{ for all } w \in W_0 \} \]

Then

\[ K_{\mathfrak{h}_d}(pt)^{\text{det}} = \{ f \in K_{\mathfrak{h}_d}(pt) \mid wf = \text{det}(w) f, \text{ for all } w \in W_0 \} \]

is a free \( K_{\mathfrak{h}_d}(pt)^{W_0} \)-module of rank 1.

\( K_{\mathfrak{h}_d}(pt)^{W_0} \) and \( K_{\mathfrak{h}_d}(pt)^{\text{det}} \) have bases

\[ m_{\lambda} = \Pi_w y^{\lambda w} \quad \text{and} \quad e_{\lambda^+\mu} = \delta y^{\lambda w}, \quad \lambda, \mu \in P^+ = \mathfrak{h}_d^{W_0} \]

where

\[ \Pi_w = \sum_{w \in W_0} w \quad \text{and} \quad \delta = \sum_{w \in W_0} \text{det}(w^{-1}) w \]
Weyl character formula

$K_{tv}(pt) W_0 \rightarrow K_{tv}(pt) \det f \rightarrow a_t f$

a naive basis

$m_{\lambda}$

$s_{\lambda} \leftarrow a_{\lambda+\rho} \text{ naive basis}$

$m_{\lambda}$ are the monomial symmetric functions

$s_{\lambda}$ are the Weyl characters or Schur functions

Note:

$K_{tv}(pt) W_0 = K_{tv}(pt) = K (G^v \text{-modules})$

and $s_{\lambda}$ are the classes of the simple modules.
Double affine Hecke algebra $\mathcal{H}$

\[ \mathcal{H} = \text{span} \{ q^k x^w x^{\nu} y^{\tau} / k \in \mathbb{Z}, w \in W_0, x^{\pm} \in \mathbb{Z} \} \]

\[ \mathcal{H}^v = \text{span} \{ x^w T_w / w \in W_0 \} \]

\[ \mathcal{H}_0 = \text{span} \{ T_w / w \in W_0 \} \]

with

\[ q^k \in \mathbb{Z}(\mathcal{H}), \quad x^w x^v = x^{w+v}, \quad y^\nu y^\tau = y^{\nu+\tau} \]

$\mathcal{H}_0$ is generated by $T_1, \ldots, T_n$ with

\[ T_i^2 = (t^i - t^{-i}) T_i + 1 \quad \text{and} \quad \overbrace{T_i T_{i+1} \cdots T_j}^{m_{ij}} = \overbrace{T_{i+1} T_i \cdots T_j}^{m_{ij}} \]

where $m_{ij} = f_i f_j$.

$\mathcal{H}^v$ has a unique $1$-dim'l module

\[ \text{span} \{ x^w \} \text{ with } T_i x^w = t^i x^w \text{ for } i = 1, \ldots, n. \]

The polynomial representation of $\mathcal{H}$ is

\[ \text{Ind}_{\mathcal{H}^v}^{\mathcal{H}}(e) = \mathcal{H}^v = \text{span} \{ q^k x^w x^{\nu} y^{\tau} / k \in \mathbb{Z}, x^{\pm} \in \mathbb{Z} \} \]

\[ = K[U(\text{pt}) \mathcal{H}] \]
Macdonald polynomials

Let $t_0, x \in H_0$ be such that

$$ t_i^0 = t_i^0 \quad \text{and} \quad t_i = (-t_i^{-1})^0 $$

for $i = 1, \ldots, n$. At $t = 1$, $t_0 = t_0$ and $x_0 = x_0$.

Then

$$ K_{\tau}(pt) \varepsilon \cdot \tilde{H} \tilde{H} = \varepsilon_0 \tilde{H} \tilde{H} = K_{\tau}(pt) \varepsilon_0 \tilde{H} $$

The nonsymmetric Macdonald polynomial

$\varepsilon \tau = \varepsilon \tau(lg)$ in $K_{\tau}(pt)$ is given by

1. $\varepsilon \tau H$ is an eigenvector of all $X^i$ (acting on $\tilde{H} \tilde{H}$)
2. $\varepsilon \tau = y^i + \text{lower stuff}$

The symmetric Macdonald polynomial

$p_{\tau} = p_{\tau}(lg)$ in $K_{\tau}(pt) W_0$ is given by

$$ p_{\tau} \tilde{H}^0 = \varepsilon \tau \tilde{H} \tilde{H}$$

Define $p_{\tau + n} = p_{\tau + n}(lg)$ in $K_{\tau}(pt)$ by

$$ p_{\tau + n} \tilde{H}^0 = \varepsilon \tau \tilde{H} \tilde{H}$$
Big picture

\[ K_{\nu}(pt) W_0 \mapsto H_0 H_0 \mapsto E H_0 \]

\[ f \mapsto P_{\nu}(q,t)f \]

\[ E_{\lambda} W_0 = P_{\nu}(q,t) W_0 \]

\[ P_{\lambda}(q,t) W_0 \mapsto P_{\lambda+\nu}(q,t) W_0 = E_{\lambda+\nu} W_0 \]

At \( q = 0 \) this picture becomes

\[ K_{\nu}(pt) W_0 \mapsto H_0 H_0 \mapsto E H_0. \]

\[ f \mapsto P_{\nu}(0,t)f \]

\[ E_{\lambda} W_0 = P_{\nu}(0,t) W_0 \]

\[ S_{\lambda} W_0 = P_{\nu}(0,0) W_0 \mapsto P_{\lambda+\nu}(0,0) W_0 = E_{\lambda+\nu} W_0 \]

where \( H = \text{span} \{ T_w Y_{\lambda^w}^a \mid w \in W_0, \lambda^w \in \Pi \} \).

At \( q = 0, t = 1 \) this becomes

\[ K_{\nu}(pt) W_0 \mapsto K_{\nu}(pt) \det \]

\[ f \mapsto a_{\nu} f \]

\[ \Pi_{\nu} Y_{\lambda}^a = m_{\lambda^\nu} \]

\[ S_{\lambda} \mapsto a_{\lambda+\nu} = E_{\nu} Y_{\lambda+\nu} \]
Remarks

(1) At $q \neq 0$, $Z(\mathbb{H})$ is trivial ($Z(\mathbb{H}) = \mathcal{O}[q^{1/2}, 1]$)
At $q = 0$, $Z(\mathbb{H})$ is big, and contains

$$K_{\mathfrak{g}(pt)}^{\omega_0} = Z(\mathbb{H}) \quad \text{(theorem of Bernstein)}.$$ 

(2) The Satake isomorphism is

$$K_{\mathfrak{g}(pt)}^{\omega_0} \xrightarrow{\gamma} \mathcal{D}_0 H \mathcal{D}_0$$

$$P_{\mathfrak{g}(pt)}(t) \xleftarrow{\iota} \mathcal{D}_0 H \mathcal{D}_0$$

and

$$P_{\mathfrak{g}(pt)}(t)$$ is the Macdonald spherical function, or

 Hague-Littlewood polynomial.

(3) $H$ is a Grothendieck ring (product is convolution)

if $I$ equiv. perverse sheaves on $G/I$

$\mathcal{D}_0 H \mathcal{D}_0$ = Groth. ring of $K$-equiv. perverse sheaves

on $G/K$.

$G/I$ = affine flag variety $G/K =$ loop Grassmanian

$\{5_{\mathfrak{g}, \mathcal{D}_0}\}$ is the Kazhdan-Lusztig basis of $\mathcal{D}_0 H \mathcal{D}_0$

(i.e. $5_{\mathfrak{g}, \mathcal{D}_0}$ is the image of $\mathcal{I}C (K_{\mathfrak{g}(pt)}^{\omega_0}, \mathbb{R}^0)$.
(4) \[ K_{\nu} (pt) \text{ det} \]
\[ \mathcal{A} \mathcal{H} \mathcal{H} \]
\[ \mathcal{A} \mathcal{H} \mathcal{H}_0 \]
are "Fock spaces"

and \[ \mathcal{A} \mathcal{H} \mathcal{H} \rightarrow \mathcal{A} \mathcal{H} \mathcal{H}_0 \] are
"boson-Fermion correspondences". The big picture
at \( q = 0 \) is a 1981 paper of Lusztig which
kicked off "Geometric Langlands".

(5) In \( \mathcal{H} \)

\[ T_i X^\mu = X^\mu + \left( t_i^+ t_i^- \right) \frac{X^\mu - X^\mu_{i+}}{1 - X^{-i}} \] (Berenstein-Lusztig relation)

is equivalent to

\[ T_i X^\mu = X^\mu_{i+} T_i \], where

\[ T_i = T_i + \frac{t_i^+ t_i^-}{1 - X^{-i}} = T_i^{-1} + \frac{(t_i^+ t_i^-) X^\mu_{i+}}{1 - X^{-i}} \] (intertwiner)

If \( Y^\mu = s_1 \ldots s_k \) is a minimal length walk
to \( Y^\mu \) in \( W \), then
in $\mathcal{A}$,

$$y^x = T_i \epsilon_1 \ldots T_i \epsilon_2 \quad \text{where}$$

$$\epsilon_x = \begin{cases} +1, & \text{if the } k \text{th step is } + \uparrow \\ -1, & \text{if the } k \text{th step is } + \downarrow \end{cases}$$

and

$$E_x^x = T_i \epsilon_1 \ldots T_i \epsilon_2$$

Using folded alcove walks this can be expanded to give a formula

$$E_x^x = \sum \text{ (explicit) } y_{\text{end}(p)} \text{ folded alcove } \text{ coeffs}$$

$$\text{ paths } p$$

which has similar coefficients to the Haglund-Haiman-Lecouvin formula for $E_x^x$ in type $GL_n$, and generalizes

$$s_{\alpha}^\nu = \sum y_{\text{wt}(p)} = \sum y_{\text{end}(p)}$$

$$\text{ column strict } \text{ tableau } \nu \text{ Littlemann } \text{ paths } \nu$$

and the tableted positively folded walks labeling points in MV intersections $\text{Im} \cap \mathcal{A} \cap \nu \cap \mathcal{I}$. 