W-algebras notes

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August 31, 2016

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1 Introduction

This is mostly lifted from Arakawa’s paper [Ar].

2 Vertex algebras, associative filtered algebras and Poisson algebras

2.1 Vertex algebras

A vertex algebra is a vector space $V$ with a linear map

$$V \longrightarrow (\text{End}(V))[z, z^{-1}]$$

$$a \mapsto a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}$$

and elements $1 \in V$ and $T \in \text{End}(V)$ such that
(a) \(1(z) = \text{id}_V\),

(b) If \(a, b \in V\) and \(a_{(-1)}1 = a\) then there exists \(\ell \in \mathbb{Z}_{>0}\) such that if \(n \in \mathbb{Z}_{>\ell}\) then \(a_{(n)}b = 0\),

(c) If \(a \in V\) then \((Ta)(z) = [T, a(z)] = \frac{d}{dz}a(z)\),

(d) If \(a, b \in V\) then there exists \(\ell \in \mathbb{Z}_{>0}\) such that if \(n \in \mathbb{Z}_{>\ell}\) then

\[ (z - w)^{n}[a(z), b(w)] = 0, \quad \text{in End}(V). \]

A **conformal vertex algebra** is a vertex algebra \(V\) with

\[ \omega \in V \quad \text{such that} \quad \omega(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \]

then there exists \(c_V \in \mathbb{C}\) such that

(e) If \(m, n \in \mathbb{Z}\) then \([L_m, L_n] = (m - n)L_{m+n} + \frac{(m^3 - m)}{12}c_V \delta_{m,-n}\),

(f) \(L_{-1} = T\), and

(g) \(L_0\) is diagonalizable on \(V\).

A **graded conformal vertex algebra** is a conformal vertex algebra \(V\) with

\[ V = \bigoplus_{d \in \frac{1}{2}\mathbb{Z}} V_d, \quad \text{where} \quad V_d = \{ a \in V \mid L_0 a = da \}. \]

Notation:

- The map \(V \to \text{End}(V)[[z, z^{-1}]]\) is the state-field correspondence.
- A **field** is an element of \(\{ a(z) \mid a \in V \}\).
- A **mode** is an element of \(\{ a_{(n)} \mid a \in V, n \in \mathbb{Z} \}\).
- The constant \(c_V\) is the central charge.
- The degree of a homogenous element \(a \in V\) is the conformal weight of \(a\).

Let \(V\) be a vertex algebra. A **\(V\)-module** is a vector space \(M\) with a linear map

\[ V \longrightarrow \text{End}(M)[[z, z^{-1}]] \]

\[ a \mapsto a^M(z) = \sum_{n \in \mathbb{Z}} a_{(n)}^M z^{-n-1} \]

such that

(a) \(1^M(z) = \text{id}_M\),

(b) If \(a \in V\) and \(m \in M\) then there exists \(\ell \in \mathbb{Z}_{>0}\) such that if \(n \in \mathbb{Z}_{>\ell}\) then \(a_{(n)}^M m = 0\).
(c) If \( p, q, r \in \mathbb{Z} \) and \( a, b, c \in \mathbb{Z} \) then, in \( \text{End}(M) \),
\[
\sum_{i \in \mathbb{Z}_{\geq 0}} \binom{p}{i} (a_{p+i} b)^{M}_{p+q+i} = \sum_{i \in \mathbb{Z}_{\geq 0}} (-1)^{i} \binom{r}{i} (a^{M}_{p+r-i} b^{M}_{q+i}) - (-1)^{r} b^{M}_{q+r+i} a^{M}_{p+i}).
\]

**Proposition 2.1.** Let \( V \) be a vertex algebra.
(a) The category of \( V \)-modules is an abelian category.
(b) \( V \) is a \( V \)-module (the adjoint module).

*Proof. Proof idea for (b):* Show that if \( a, b \in V \) and \( p, q, r \in \mathbb{Z} \) then, in \( \text{End}(M) \),
\[
\sum_{i \in \mathbb{Z}_{\geq 0}} \binom{p}{i} (a_{p+i} b)^{M}_{p+q+i} = \sum_{i \in \mathbb{Z}_{\geq 0}} (-1)^{i} \binom{r}{i} (a^{M}_{p+r-i} b^{M}_{q+i}) - (-1)^{r} b^{M}_{q+r+i} a^{M}_{p+i}).
\]

Let \( V \) be a graded conformal vertex algebra.

- \( V \) is *rational*, or (representation) *semisimple*, if every \( V \)-module is completely reducible.

### 2.2 The enveloping algebra \( U(V) \) of \( V \)

Let \( V \) be a graded conformal vertex algebra.
\[
V = \bigoplus_{d \in \frac{1}{2} \mathbb{Z}} V_{d}, \quad \text{where} \quad V_{d} = \{ a \in V \mid L_{0} a = da \}.
\]

For homogeneous \( a, b \in V \) define
\[
a \circ b = \sum_{i \in \mathbb{Z}_{\geq 0}} \binom{\text{wt}(a)}{i} a_{(i-2)} b, \quad \text{and} \quad a \ast b = \sum_{i \in \mathbb{Z}_{\geq 0}} \binom{\text{wt}(a)}{i} a_{(i-1)} b.
\]

The *enveloping algebra of \( V \), or \((L_0\text{-twisted}) Zhu’s algebra of \( V \)* is
\[
U(V) = \frac{V}{O(V)}, \quad \text{where} \quad O(V) = \mathbb{C}\text{-span}\{ a \circ b \mid \text{homogeneous } a, b \in V \},
\]

with product
\[
U(V) \otimes U(V) \rightarrow U(V) \quad (a, b) \mapsto a \ast b.
\]

Define a filtration on \( U(V) \) by
\[
F_{d} U(V) = \text{(image of } V_{\leq d} \text{ in } U(V)).
\]
Proposition 2.2. The map $\pi_P: \text{Ps}(V) \to \text{gr}_F U(V)$ given by

$$\pi_P(a + C_2(V)_p) = (a + (O(V) \cap V_{\leq p})) + V_{\leq (p - \frac{1}{2})},$$

for $a \in \text{Ps}(V)_p$, is a surjective homomorphism of graded Poisson algebras.

Let $V$ be a vertex algebra and let $M$ be a graded $V$-module. For homogeneous $a \in V$ and $m \in M$ define

$$a \circ m = \sum_{i \in \mathbb{Z}_{\geq 0}} \left( \frac{\text{wt}(a)}{i} \right) a_{(i-2)}^M m.$$

Let

$$O(M) = \mathbb{C}\text{-span}\{a \circ m \mid \text{homogeneous } a \in V \text{ and } m \in M\}.$$

Define

$$U(M) = \frac{M}{O(M)},$$

with $U(V)$-bimodule structure given by

$$a * m = \sum_{i \in \mathbb{Z}_{\geq 0}} \left( \frac{\text{wt}(a)}{i} \right) a_{(i-1)}^M m \quad \text{and} \quad m * a = \sum_{i \in \mathbb{Z}_{\geq 0}} \left( \frac{\text{wt}(a) - 1}{i} \right) a_{(i-1)}^M m.$$

Theorem 2.3. The functor

$$V\text{-Mod} \longrightarrow U(V)\text{-biMod}$$

is a right exact functor.

2.3 The Poisson algebra $\text{Ps}(V)$ of $V$

Let $V$ be a graded conformal vertex algebra

$$V = \bigoplus_{d \in \frac{1}{2} \mathbb{Z}} V_d,$$

where $V_d = \{ v \in V \mid L_0 v = dv \}$.

Let

$$C_2(V) = \mathbb{C}\text{-span}\{a_{(-2)} v \mid v \in V\}.$$

The Poisson algebra of $V$, or Zhu’s $C_2$-algebra of $V$, is

$$\text{Ps}(V) = \frac{V}{C_2(V)}$$

with $\bar{a} \cdot \bar{b} = \bar{a_{(-2)} b}$ and $\{\bar{a}, \bar{b}\} = \bar{a_{(0)} b}$.

and grading

$$\text{Ps}(V) = \bigoplus_{d \in \frac{1}{2} \mathbb{Z}} \text{Ps}(V)_d,$$

where $\text{Ps}(V)_d = \text{image of } V_d \text{ in } \text{Ps}(V)$.

Proposition 2.4. Let $V$ be a graded conformal vertex algebra. Then $\text{Ps}(V)$ is a graded Poisson algebra.
Let $V$ be a graded conformal vertex algebra.

- $V$ is finitely strongly generated if $\text{Ps}(V)$ is a finitely generated ring.
- $V$ is $C_2$-cofinite, or lisse, if $\text{Ps}(V)$ is a finite dimensional.

**Proposition 2.5.** Let $V$ be a graded conformal vertex algebra and let $M$ be a graded $V$-module. Define
\[ C_2(M) = \mathbb{C}\text{-span}\{a^M_{(-2)}m \mid a \in V, m \in M\}. \]
Then
\[ \text{Ps}(M) = \frac{M}{C_2(M)}, \quad \text{with} \quad \overline{a} \cdot \overline{m} = a^M_{(-1)}m \quad \text{and} \quad \{\overline{a}, \overline{m}\} = a^M_{(0)}m. \]
is a Poisson module for $\text{Ps}(V)$.

### 2.4 Poisson algebras and modules

A Poisson algebra is a commutative $\mathbb{C}$-algebra $R$ with a bilinear map
\[ R \otimes R \rightarrow R \quad (r_1, r_2) \mapsto \{r_1, r_2\} \quad \text{such that} \]

(a) If $a, b \in R$ then $\{a, b\} = -\{b, a\}$,
(b) If $a, b, c \in R$ then $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0$,
(c) If $a, b, c \in R$ then $\{a, bc\} = \{a, b\}c + b\{a, c\}$.

Let $R$ be a Poisson algebra (really we should use graded Poisson superalgebras). A Poisson module for $R$ is an $R$-module $M$ with a bilinear map
\[ R \otimes M \rightarrow M \quad (r, m) \mapsto \{r, m\} \quad \text{such that} \]

(a) If $r_1, r_2 \in R$ then $\{r_1, r_2\}m = r_1r_2m - r_2r_1m$,
(b) If $r_1, r_2 \in R$ and $m \in M$ then $\{r_1, r_2m\} = \{r_1, r_2\}m + r_2\{r_1, m\}$.
(c) If $r_1, r_2 \in R$ and $M \in M$ then $\{r_1r_2, m\} = r_1\{r_2, m\} + r_2\{r_1, m\}$.

**Notation:**

- $R\text{-PMod}$ is the category of Poisson modules for $R$.  

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2.5 Associative filtered algebras

An associative filtered algebra is a $C$-algebra $U$ with a filtration

$$C = U_0 \subseteq U_1 \subseteq \cdots \text{ such that } \bigcup_{i \in \mathbb{Z}_{\geq 0}} U_i = U \text{ and } U_i U_j \subseteq U_{i+j}.$$ 

WHAT WE REALLY NEED IS INCREASING EXHAUSTIVE SEPARATED FILTRATION. WHAT DOES THIS MEAN?? Notation:

- $A$-biMod is the category of $A$-bimodules.

Let $A$ be an associative filtered algebra and let $M \in A$-biMod. A compatible filtration is a $\mathbb{Z}$-filtration on $M$ such that if $p, q \in \mathbb{Z}$ then

$$(F_p U) \cdot (F_q M) \subseteq F_{p+q} M, \quad (F_q M) \cdot (F_p A) \subseteq F_{p+q} M, \quad \text{and } [F_p A, F_q M] \subseteq F_{p+q-1} M.$$ 

**Proposition 2.6.** Let $U$ be an associative filtered algebra.

(a) Then

$$\text{gr}\ F U = \bigoplus_{p \in \frac{1}{2} \mathbb{Z}} \frac{F_p U}{F_{p-rac{1}{2}} U} \quad \text{with} \quad \{\bar{a}, \bar{b}\} = ab - ba,$$

is a graded Poisson algebra.

(b) Let $M \in A$-biMod with a compatible filtration. Then

$$\text{gr}\ F M = \bigoplus_{p \in \frac{1}{2} \mathbb{Z}} \left( \frac{F_p M}{F_{p-rac{1}{2}} M} \right) \quad \text{is a graded Poisson module for } \text{gr}\ F U.$$ 

3 The vertex algebra $V^k(g)$

Let $g$ be a finite dimensional simple Lie algebra with nondegenerate ad-invariant inner product $(\cdot | \cdot) : g \otimes g \to \mathbb{C}$. Let

$$\hat{g} = g[t, t^{-1}] \oplus \mathbb{C}K, \quad \text{with } [K, xt^m] = 0, \quad \text{and}$$

$$[xt^m, yt^n] = [x, y]t^{m+n} + m(x|y)\delta_{m, -n} K,$$

for $x, y \in g$ and $m, n \in \mathbb{Z}$. Let $k \in \mathbb{C}$. The universal affine vertex algebra associated to $g$ at level $k$ is the vector space

$$V^k(g) = U\hat{g} \otimes_{U(g[t] \oplus \mathbb{C}K)} \mathbb{C}k,$$

where $\mathbb{C}k = \mathbb{C}\text{-span}\{v\}$ with $Kv = kv$ and $xt^mv = 0$ for $x \in g$ and $m \in \mathbb{Z}$, and $V^k(g)$ has vertex algebra structure determined by

$$1 = v, \quad (xt^{-1}1)(z) = \sum_{n \in \mathbb{Z}} (xt^n)z^{-n-1}, \quad \text{for } x \in g,$$

and, if $\{X_1, \ldots, X_{\ell}\}$ is a basis of $g$ and $\{X^1, \ldots, X^n\}$ is the dual basis of $g$ with respect to $(\cdot | \cdot)$ then

$$\omega = \frac{1}{2(k + h^\vee)} \sum_{i=1}^\ell (X_i t^{-1})(X^i t^{-1})1.$$

(the Sugawara vector)
3.1 The associative filtered algebra of $V^k(g)$

Let $U_g$ be the enveloping algebra of $g$. The PBW filtration on $U_g$ is given by

$$f_{-1}U_g = 0, \quad F_0U_g = \mathbb{C}, \quad F_pU_g = g \cdot (F_{p-1}U_g) + (F_{p-1}g).$$

The Poincaré-Birkhoff-Witt theorem gives that

$$\text{gr}_pU_g \cong S(g) = \mathbb{C}[g^*].$$

Let $f$ be a nilpotent element of $g$. The Kazhdan filtration of $U_g$ with respect to $f$ is given by

$$K_pU_g = \sum_{i-j \leq p} F_iU_g[j], \quad \text{where} \quad F_pU_g[j] = \{ u \in F_pU_g | \text{ad}(h)(u) = 2ju \}.$$  

**Proposition 3.1.** Let $f$ be a nilpotent element of $g$ and let

$$K_0U_g \subseteq K_1U_g \subseteq K_2U_g \subseteq \cdots$$

be the Kazhdan filtration of $U_g$ with respect to $f$. Then

$$\text{gr}_KU_g \cong S(g) = \mathbb{C}[g^*].$$

IS THE PBW FILTRATION THE KAZHDAN FILTRATION OF $U_g$ with respect to the regular nilpotent????

**Proposition 3.2.** (Frenkel-Zhu) (equation (25) in Arakawa)

(a) $U(V^k(g)) \cong U_g$.

(b) The filtration on $U(V^k(g))$ given by

$$F_p(U(V^k(g))) = (\text{image of } V^k(g)_{\leq p} \text{ in } U(V^k(g)))$$

corresponds to the PBW??? or Kazhdan??? filtration on $U_g$.

For a $U_g$-bimodule $M$ define $\text{ad} : g \rightarrow \text{End}(M)$ by

$$\text{ad}(x)m = xm - mx, \quad \text{for } x \in g \text{ and } m \in M.$$  

The action of $g$ by $\text{ad}$ is the adjoint action of $g$ on $M$.

• $U_g$-biMod is the category of $U_g$-bimodules.

• $\mathcal{H}C$ is the full subcategory of $U_g$-biMod consisting of $M$ such that

  the ad action of $g$ on $M$ is locally finite.

**Proposition 3.3.**

(a) $U : \text{KL}_k \rightarrow \mathcal{H}C$ is a right exact functor.

(b) If $M \in \text{KL}_k$ and $M$ is finitely generated then $U(M)$ is a finitely generated $U_g$ module.

**Proposition 3.4.**

(a) $V^k(g) \in \text{KL}^\Delta_k$.

(b) $\text{KL}^\Delta_k = \{ m \in \text{KL}_k | M \text{ is a free } U(g[t^{-1}]t^{-1})\text{-module of finite rank} \}$.

(c) If $M \in \text{KL}^\Delta_k$ then $\text{Ps}(M) \cong \text{gr}_pU(M)$.

(d) $U : \text{KL}^\Delta_k \rightarrow \mathcal{H}C$ is an exact functor.
3.2 The Poisson algebra of $V^k(\mathfrak{g})$

The commutative algebra $\mathbb{C}[\mathfrak{g}^*] = S(\mathfrak{g})$ is a Poisson algebra with the Kirillov-Kostant Poisson bracket.

**Proposition 3.5.**

(a) $C_2(V^k(\mathfrak{g})) = \mathfrak{g}[t^{-1}]t^{-2}V^k(\mathfrak{g})$.

(b) The map

$$\Phi: \mathbb{C}[\mathfrak{g}^*] \to \text{Ps}(V^k(\mathfrak{g}))$$

determined by $\Phi(x) = (xt^{-1})I$ for $x \in \mathfrak{g}$,

is an isomorphism of Poisson algebras.

A $\mathbb{C}[\mathfrak{g}^*]$-Poisson module is a $\mathbb{C}[\mathfrak{g}^*]$-module $M$ with a linear map $\text{ad}: \mathfrak{g} \to \text{End}(M)$ such that

(a) If $x, y \in \mathfrak{g}$ then $\text{ad}(\{x, y\}) = \text{ad}(x)\text{ad}(y) - \text{ad}(y)\text{ad}(x)$,

(b) If $x \in \mathfrak{g}$, $f \in \mathbb{C}[\mathfrak{g}^*]$ and $m \in M$ then

$$\text{ad}(x)(fm) = \{x, f\}m + f\text{ad}(x)(m).$$

The action of $\mathfrak{g}$ by $\text{ad}$ is the adjoint action of $\mathfrak{g}$ on $M$.

- $\mathbb{C}[\mathfrak{g}^*]$-$\text{PMod}$ is the category of $\mathbb{C}[\mathfrak{g}^*]$-Poisson modules.
- $\mathcal{H}\mathcal{C}$ is the full subcategory of $\mathbb{C}[\mathfrak{g}^*]$-$\text{PMod}$ consisting of $M$ such that the $\text{ad}$ action of $\mathfrak{g}$ on $M$ is locally finite.

3.3 $V^k(\mathfrak{g})$-modules

A smooth $\hat{\mathfrak{g}}$-module is a $\hat{\mathfrak{g}}$-module $M$ such that if $m \in M$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq \ell}$ and $x \in \mathfrak{g}$ then $(xt^n)m = 0$.

**Proposition 3.6.** $A V^k(\mathfrak{g})$-module is the same thing as a smooth $\hat{\mathfrak{g}}$-module of level $k$.

View $\mathfrak{g}$ as a Lie subalgebra of $\hat{\mathfrak{g}}$ by the inclusion $x \mapsto xt^0$.

- $V^k(\mathfrak{g})$-$\text{Mod}$ is the abelian category of $V^k(\mathfrak{g})$-modules.
- $V^k(\mathfrak{g})$-$\text{gMod}$ is the full subcategory of $V^k(\mathfrak{g})$-$\text{Mod}$ of positively graded $V^k(\mathfrak{g})$-modules.
- $\text{KL}_k$ is the full subcategory of graded $V^k(\mathfrak{g})$-modules $M$ such that $\mathfrak{g}$ acts locally finitely on $M$. 


• $\text{KL}_k^\Delta$ is the full subcategory of $\text{KL}_k$-modules $M$ which satisfy: there exists $r \in \mathbb{Z}_{\geq 0}$ and

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_r = M \quad \text{with} \quad \frac{M_i}{M_{i-1}} \cong \Delta_i^g(E_i),$$

for finite dimensional $g$-modules $E_1, \ldots, E_r$.

**Proposition 3.7.** Let $M$ be a $V^k(g)$-module.

(a) $C_2(M) = g[t^{-1}]t^{-2}M$.

(b) $\text{Ps}(M) = \frac{M}{g[t^{-1}]t^{-2}M}$ is a Poisson module for $\mathbb{C}[g^*]$ with

$$x \cdot m = (xt^{-1})m \quad \text{and} \quad \{x, m\} = (xt^0)m, \quad \text{for} \ x \in g \ \text{and} \ m \in M.$$

**References**