Existence of limits

Example: What is \( \lim_{x \to 0} \frac{1}{x} \)?

- If \( x = 1 \) then \( \frac{1}{x} = 10 \)
- If \( x = 0.01 \) then \( \frac{1}{x} = 100 \)
- If \( x = 0.001 \) then \( \frac{1}{x} = 1000 \)
- If \( x = 0.0001 \) then \( \frac{1}{x} = 10000 \)

So, it looks like \( \lim_{x \to 0} \frac{1}{x} = \infty \)

- If \( x = -1 \) then \( \frac{1}{x} = -10 \)
- If \( x = -0.01 \) then \( \frac{1}{x} = -100 \)
- If \( x = -0.001 \) then \( \frac{1}{x} = -1000 \)

So, it looks like \( \lim_{x \to 0} \frac{1}{x} = -\infty \).

Since \( \lim_{x \to 0^+} \frac{1}{x} = \infty \) and \( \lim_{x \to 0^-} \frac{1}{x} = -\infty \),

\[ \lim_{x \to 0} \frac{1}{x} = \text{undefined} \]

Example: \( \lim_{x \to 1} \ln x = \) ??

Look at the graph of \( \ln x \).

Notes:
- \( e^{0.2} \approx 1.2214 \)
- \( e^{0.5} \approx 1.6487 \)
- \( e^{1} = e \approx 2.71828 \)
- Note: \( y = \ln x \) means \( e^{y} = x \).

So, this graph is the same as the left one but with \( x \) and \( y \) switched.

So, from the graph, \( \ln x \) doesn't even make sense for \( x \) close to 0. So

\[ \lim_{x \to 0} \ln x \] is certainly undefined.

Note: If we allow \( x \) to get closer and closer to 1 and be a complex number then

\( \ln -1 = i\pi \) and \( i3\pi \) and \( i5\pi \)...

Since \( e^{i\pi} = \cos \pi + i\sin \pi = -1 + 0 = -1 \) and \( \ln -1 = i\pi \).

Still \( \lim_{x \to 1} \ln x \) is undefined since it can't be \( i\pi \) and \( 3i\pi \) and \( 5i\pi \) ... all at once.
Example $\lim_{x \to \infty} \sin x$

The graph of $\sin x$ is

So, as $x$ gets larger and larger, $\sin x$ keeps going back and forth between $-1$ and $+1$.
So $\sin x$ doesn't get closer and closer to anything as $x$ gets larger and larger.

So $\lim_{x \to \infty} \sin x$ is undefined.

Continuous functions

A function is continuous if $f(x)$ doesn't jump when $x$ changes. The function $f(x)$ is not continuous exactly at the places where it jumps.

A function $f(x)$ is **continuous at** $x = a$ if it doesn't jump at $x = a$,

i.e. if $\lim_{x \to a} f(x) = f(a)$

Example $f(x) = 1/x$, Round down function

$\lfloor 3.2 \rfloor = 3.$

$f(x)$ is continuous if $x \neq 0, \pm 1, \pm 2, \pm 3, \ldots$

Note: $\lim_{x \to 0^+} 1/x = \infty$ and $\lim_{x \to 0^-} 1/x = -\infty$.

$f(x) = \lfloor x \rfloor$ is the **round up function**

$\lfloor 3.2 \rfloor = 4.$
Example: 
\[ f(x) = \begin{cases} 
 1 + x^2, & 0 \leq x < 1, \\
 2 - x, & x \geq 1. 
\end{cases} \]

\( f(x) \) jumps at \( x = 1 \).

\[ \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 1 + x^2 = 2. \]

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 2 - x = 1. \]

So \( \lim_{x \to 1} f(x) \) is UNDEFINED.

Example: 
\[ f(x) = \begin{cases} 
 \frac{\sin 3x}{x}, & x \neq 0 \\
 1, & x = 0 
\end{cases} \]

\( \sin 3x \) is continuous everywhere and \( x \) is continuous everywhere.

So \( \frac{\sin 3x}{x} \) is continuous everywhere.

EXCEPT, it makes no sense when \( x = 0 \).

Now what is happening when \( x = 0 \)?

\[ \lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot 3 = 1 \cdot 3 = 3. \]

BUT \( f(0) = 1 \),

So \( \lim_{x \to 0} f(x) \neq f(0) \) in this case.

So \( f(x) \) is not continuous when \( x = 0 \).