A function $f(x)$ is continuous at $x = a$ if it doesn't jump at $x = a$,

\[ \lim_{x \to a} f(x) = f(a) \]

Not continuous at $x = a$.

Think about

\[ \frac{df}{dx}_{x = a} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} \]

in terms of the graph.

\[ \frac{f(a+\Delta x) - f(a)}{\Delta x} = \frac{\text{change in } f}{\text{change in } x} = \frac{\text{rise}}{\text{run}} = \text{slope of line connecting } (a, f(a)) \text{ and } (a+\Delta x, f(a+\Delta x)) \]

\[ \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = \text{slope of } f \text{ at the point } x = a. \]

A function $f(x)$ is differentiable at $x = a$ if the derivative $\frac{df}{dx}_{x = a}$ exists,

\[ \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

ie. if the slope of the graph of $f(x)$ at $x = a$ exists.

Example: Graph $f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Then

\[ \frac{df}{dx}_{x = a} = \begin{cases} 1, & \text{if } a > 0 \\ -1, & \text{if } a < 0 \\ \text{does not exist, if } a = 0 \end{cases} \]

So $f$ is not differentiable at $x = 0$.

Example: Graph $y = x^{\frac{1}{3}}$.

Notes:

(a) $y = x^{\frac{1}{3}}$ is the same as $y^3 = x$. 

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Notes:
\[ \frac{dy}{dx} = \frac{1}{3} x^{-\frac{2}{3}} \]

So \( \frac{dy}{dx} \Big|_{x=0} = \infty \). So \( f(x) \) is not differentiable at \( x=0 \).

A function \( f(x) \) is increasing if it is going up at \( x=a \),

i.e. if \( f(a+dx) > f(a) \) for all small \( dx > 0 \),

i.e. if \( \frac{df}{dx} \Big|_{x=a} > 0 \).

\[ y = f(x) \]

\[ f(x) \]

\[ f(x) \]

\[ f(x) \]

\[ f(x) \]

\[ f(x) \]

A function \( f(x) \) is decreasing if it is going down at \( x=a \),

i.e. if \( f(a+dx) < f(a) \) for all small \( dx > 0 \),

i.e. if \( \frac{df}{dx} \Big|_{x=a} < 0 \).

A point of inflection is a point where \( f \) changes from concave up to concave down, or from concave down to concave up.
A local maximum is a point \( x = a \) where \( f(x) \) is larger than the \( f(x) \) around it.

A local minimum is a point \( x = a \) where \( f(x) \) is smaller than the \( f(x) \) around it.

i.e. \( f(a) < f(a + \Delta x) \) for small \( \Delta x \).

\[ \text{local maximum at } x = a_1 \]
\[ \text{local maximum at } x = a_2 \]
\[ \text{local maximum at } x = a_3 \]
\[ \text{local minimum at } x = a_4 \]

A critical point is a point where a maximum or minimum might occur.

Note:

1) If \( f(x) \) is continuous and differentiable and \( x = a \) is a maximum then

\[ \frac{df}{dx}\bigg|_{x = a} = 0 \quad \text{and} \quad \frac{d^2f}{dx^2}\bigg|_{x = a} < 0 \]

2) If \( f(x) \) is continuous at \( x = a \), \( f(x) \) is differentiable at \( x = a \),

\[ \frac{df}{dx}\bigg|_{x = a} = 0 \quad \text{and} \quad \frac{d^2f}{dx^2}\bigg|_{x = a} > 0 \]

\[ x = a \text{ is a minimum.} \]

Where can a maximum or minimum occur?

a) A point \( x = a \) where

\( f(x) \) is differentiable and \( \frac{df}{dx}\bigg|_{x = a} = 0 \).

b) A point \( x = a \) where

\( f(x) \) is not continuous.

\[ f(x) = \begin{cases} 
  x^2 + 1, & \text{if } 0 \leq x \leq 1, \\
  2 - x, & \text{if } x > 1.
\end{cases} \]

\[ x = 1 \text{ is a maximum.} \]