Applications of exponential functions

Example: If the bacteria in a culture increase continuously at a rate proportional to the number present, and the initial number is \(N_0\), find the number at time \(t\).

Idea: Change in bacteria is proportional to the amount of bacteria

\[
\frac{dB}{dt} = kB.
\]

What could \(B\) be?

\[
\frac{dB}{B} = kdt. \quad \text{So} \quad \int \frac{1}{B} dB = \int kdt.
\]

So \(\ln B = kt + C\).

So \(B = e^{kt+C} = e^C e^{kt} = Ce^{kt}\), where \(C\) is a constant.

At time \(t=0\), \(B = N_0 = Ce^{k\cdot0} = C\). So \(C = N_0\).

So \(B = N_0 e^{kt}\).

Example: A roast turkey is taken from an oven when its temperature reaches 185°F and is placed on a table in a room where the temperature is 75°F. It cools at a rate proportional to the difference between its current temperature and the room temperature.

(a) After half an hour what is the temperature?

(b) When will the turkey have cooled to 100°F?

Idea: change in temperature is proportional to current temperature - room temperature.

\[
\frac{dT}{dt} = k(T-R).
\]

So \(\frac{dT}{T-R} = kdt. \quad \text{So} \quad \int \frac{dT}{T-R} = \int kdt\)

So \(\ln(T-R) = kt + C\).

So \(T-R = e^{kt+C} = e^C e^{kt} = Ce^{kt}\),

where \(C\) is a constant.
\[ T = Ce^{kt} + R. \]

At \( t = 0 \), \( T = 185 = Ce^{k \cdot 0} + 75 = C + 75 \)

\[ C = 185 - 75 = 110. \]

\[ T = 110e^{kt} + 75. \]

At \( t = \frac{1}{2} \), \( T = 110e^{k \cdot \frac{1}{2}} + 75 = 150. \)

\[ e^{k \cdot \frac{1}{2}} = \frac{150 - 75}{110} = \frac{75}{110} \]

\[ \frac{1}{2}k = \ln \left( \frac{75}{110} \right). \]

\[ k = 2\ln \left( \frac{75}{110} \right). \]

\[ T = 110e^{2\ln \left( \frac{75}{110} \right)t} + 75. \]

At \( t = \frac{3}{4} \), \( T = 110e^{2\ln \left( \frac{75}{110} \right) \cdot \frac{3}{4}} + 75 = 110e^{\ln \left( \frac{75}{110} \right) \cdot \frac{3}{4}} + 75 \]

\[ = \frac{110 \left( \frac{75}{110} \right)^{\frac{3}{4}}}{110} + 75 = 110 \left( \frac{75}{110} \right)^{\frac{3}{4}} + 75. \]

If \( T = 100 \) then \( 110e^{2\ln \left( \frac{75}{110} \right)t} + 75 = 100 \)

\[ e^{2\ln \left( \frac{75}{110} \right)t} = \frac{100 - 75}{110} = \frac{25}{110} \]

\[ 2\ln \left( \frac{75}{110} \right)t = \ln \left( \frac{25}{110} \right). \]

\[ t = \frac{\ln \left( \frac{25}{110} \right)}{2\ln \left( \frac{75}{110} \right)}. \]

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Example: The majority of naturally occurring rhenium is \(^{187}\text{Re}\), which is radioactive and has a half-life of \(7 \times 10^9\) years. In how many years will 5% of the earth's \(^{187}\text{Re}\) decompose?

The change in rhenium is proportional to the existing amount of \(^{187}\text{Re}\).

\[ \frac{dR}{dt} = kR. \]

\[ \frac{dR}{R} = kt. \]

\[ \int \frac{dR}{R} = \int kdt. \]

\[ \ln R = kt + C. \]

\[ R = e^{kt + C} = e^{kt}e^C = Ce^{kt} \]

where \( C \) is a constant.

When \( t = 0 \) the amount is \( R_0 \). \( R_0 = Ce^{k \cdot 0} = C. \)

When \( t = 7 \times 10^9 \) the amount is \( \frac{1}{2}R_0 \).

\[ \frac{1}{2}R_0 = R_0e^{k \cdot 7 \times 10^9}. \]

\[ \frac{1}{2} = e^{k \cdot 7 \times 10^9}. \]

\[ \ln \frac{1}{2} = k \cdot 7 \times 10^9. \]

\[ k = \frac{\ln \left( \frac{1}{2} \right)}{7 \times 10^9}. \]

\[ R = R_0e^{\frac{\ln \left( \frac{1}{2} \right)}{7 \times 10^9}t}. \]
We want to know when \( R = 0.05 R_0. \)

\[
0.05 R = R_0 e^{\frac{\ln(10)}{7.4 \times 10^5} t}
\]

\[
\therefore \quad t = 7.4 \times 10^5 \ln \left( \frac{10}{10} \right) \quad \therefore \quad \ln \left( \frac{10}{10} \right) = \ln \left( \frac{10}{10} \right) = \ln \left( \frac{10}{10} \right)
\]

\[
\therefore \quad t = 7.4 \times 10^5 \ln \left( \frac{10}{10} \right) \quad \therefore \quad \ln \left( \frac{10}{10} \right) \quad \therefore \quad \ln \left( \frac{10}{10} \right)
\]

**Example:** If you buy a $200,000 home and put 10% down and take out a 30-year fixed rate mortgage at 8% per year, compute how much your payment would be if you paid it all off in one big payment at the end of 30 years.

Idea: Change in the money is .08 of its current amount.

\[
\frac{dM}{dt} = 0.08 M
\]

\[
\therefore \quad \frac{dM}{M} = 0.08 \, dt \quad \therefore \quad \int \frac{dM}{M} = \int 0.08 \, dt
\]

\[
\therefore \quad \ln M = 0.08 t + C \quad \therefore \quad M = e^{0.08 t + C} = e^{0.08 t} \cdot C
\]

where \( C \) is a constant.

\[
\therefore \quad M = C e^{0.08 t}
\]

At time \( t = 0 \) we owe $200,000 - 20,000 = 180,000.

\[
180,000 = C e^{0.08 \cdot 0} = C
\]

\[
C = 180,000
\]

After 30 years we owe

\[
M = 180,000 e^{0.08 \cdot 30} = 180,000 e^{2.4} \text{ dollars}
\]

**Example:** If you borrow $500 on your credit card at 14% interest find the amounts due at the end of 2 years if the interest is compounded

(a) annually
(b) quarterly
(c) monthly
(d) daily
(e) hourly
(f) every second
(g) every nanosecond
(h) continuously.
You own:
(a) \(500 + 500(1.14) = 500(1 + 1.14)\) after one year.
\(500(1 + 1.14)(1 + 1.14)\) after two years.
(b) \(500 + 500(1.14)^{1/4} = 500(1 + 1.14)^{1/4}\) after one quarter.
\(500(1 + 1.14)^{1/4}\) after two quarters.
\(500(1 + 1.14)^{1/4}\) after two years (8 quarters).
(c) \(500 + 500\left(\frac{1.14}{10}\right) = 500(1 + 1.14)\) after 1 month.
\(500(1 + 1.14)\) after two years (24 months).
(d) \(500 + 500\left(\frac{1.14}{365}\right) = 500\left(1 + \frac{1.14}{365}\right)\) after 1 day.
\(500\left(1 + \frac{1.14}{365}\right)^{2.365}\) after two years (2.365 days).
(e) \(500 + 500\left(\frac{1.14}{365/12}\right) = 500\left(1 + \frac{1.14}{365/12}\right)\) after 1 hour.
\(500\left(1 + \frac{1.14}{365/12}\right)^{2.365/12}\) after two years.
(f) \(500 + 500\left(\frac{1.14}{365/12/3600}\right) = 500\left(1 + \frac{1.14}{365/12/3600}\right)\) after 1 second.
\(500\left(1 + \frac{1.14}{365/12/3600}\right)^{2.365/12/3600}\) after two years.

(b) \(\lim_{n \to \infty} 500 \left(1 + \frac{1.14}{n}\right)^{n^2}\)
\(= \lim_{n \to \infty} 500 \left(1 + \frac{1.14}{n}\right)^{n^2} = 500 (e^{1.14})^n = 500 e^{2.88}\)
after two years, since
\(\lim_{n \to \infty} \left(1 + \frac{1.14}{n}\right)^n = \lim_{n \to \infty} \left(e^{\ln\left(1 + \frac{1.14}{n}\right)}\right)^n\)
\(= \lim_{n \to \infty} e^{n \ln\left(1 + \frac{1.14}{n}\right)} = \lim_{n \to \infty} e^{\left(1 + \frac{1.14}{n}\right)^n} = e^{1.14} = e^{2.88}\)

Example: A sample of a wooden artifact from an Egyptian tomb has a \(^{14}C/^{12}C\) ratio which is 54.2% of that of freshly cut wood. In approximately what year was the old wood cut? The half-life of \(^{14}C\) is 5720 years.

Idea: The change in \(^{14}C\) is proportional to the existing amount.
\(\frac{d\,^{14}C}{dt} = k\,^{14}C\).
\[ \frac{d}{dt} \ln C = k \, dt. \quad \text{So} \quad \int \frac{d}{dt} \ln C = \int k \, dt. \]

\[ \ln C = kt + c. \quad \text{So} \quad C = e^{kt+c} = e^c \cdot e^{kt} = Ke^{kt}, \]
where \( K \) is a constant.

Suppose that at \( t=0 \), the amount of \( C \) is \( C_0 \).

Then \( C_0 = K e^{k \cdot 0} = K \)

So \( C = C_0 e^{kt} \).

The half-life of \( ^{14}C \) is 5720 years. So, at \( t = 5720 \)

\[ \frac{1}{2} C_0 = C_0 e^{k \cdot 5720} \]

So \[ \frac{1}{2} = e^{k \cdot 5720} \quad \text{So} \quad \ln \left( \frac{1}{2} \right) = k \cdot 5720. \]

So \[ k = \frac{\ln \left( \frac{1}{2} \right)}{5720} \quad \text{So} \quad C = C_0 e^{\frac{\ln \left( \frac{1}{2} \right)}{5720} t} \]

Now there is 54.2\% of the original \( ^{14}C \). So \( \frac{1}{1.542} C_0 = C_0 e^{\frac{\ln \left( \frac{1}{1.542} \right)}{5720} t} \quad \text{So} \quad 0.542 = e^{\frac{\ln \left( \frac{1}{1.542} \right)}{5720} t} \]

So \[ \ln (0.542) = \frac{\ln (t)}{5720} \quad \text{So} \quad t = \frac{\ln (0.542)}{\ln (\frac{1}{1.542})} \times 5720. \]