MATH 221, Lecture 40, December 15, 2000

Helpful techniques

1) Multiplying out
2) Common denominator
3) Multiply top and bottom by the same thing
4) Completing the square
5) Change to a single Stuff
6) Change $x = \cos \theta$ to work with circles of radius 1
7) Multiply by the conjugate
   a) to divide complex numbers
   b) to get rid of radicals added together
   c) to deal with some integrals
8) Change messy trig functions to sines and cosines
9) If it's not how you want it, make it like you want it (in such a way that it is still equal to what it was before)
10) Don't panic, just write one tiny step at a time.

Remarks for review:

1) The word "prove" is the same as "explain why".
   A problem that begins with the words "Prove that" or "Show that" or "Explain why" is exactly the same as a problem with the answer given.

2) Unsimplifications for integrals:
   \[
   \cos^3 x = \frac{1}{2} (\cos x + \cos 3x) = \frac{1}{2} (1 + \cos 2x),
   \]
   \[
   \sin^3 x = \frac{1}{2} (\sin x + \sin 3x) = \frac{1}{2} (1 - \cos 2x),
   \]
   \[
   \tan x = \frac{\sin^2 x}{\cos^2 x} = \frac{1 - \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - 1,
   \]
   \[
   \cot x = \frac{\cos^2 x}{\sin^2 x} = \frac{1 - \sin^2 x}{\sin^2 x} = \frac{1}{\sin^2 x} - 1.
   \]

3) The "theory" problems were (more or less) all done in class and so they could be called "regurgitation" problems. These are:
   HW1 B1-11, HW2 A1-13, HW3 A1-31, HW3B1-6, HW3C1-3, HW4 B1-9, HW5D1-8, HW7D1-3, HW10B1-7, HW12B1-5, HW12D1-5.

These problems are the basis for the concepts in Math 221.
Things to memorize for the exam for speed.

(1) Favorite derivatives:
\[
\begin{align*}
\frac{d}{dx} \sin x &= \cos x, & \frac{d}{dx} \cos x &= -\sin x, & \frac{d}{dx} \tan x &= \sec^2 x, \\
\frac{d}{dx} \sec x &= \sec x \tan x, & \frac{d}{dx} \csc x &= -\csc x \cot x, & \frac{d}{dx} \cot x &= -\csc^2 x, \\
\frac{d}{dx} e^x &= e^x, & \frac{d}{dx} \ln x &= \frac{1}{x}. \\
\frac{d}{dx} \sin^{-1} x &= \frac{1}{\sqrt{1-x^2}}, & \frac{d}{dx} \cos^{-1} x &= -\frac{1}{\sqrt{1-x^2}}, & \frac{d}{dx} \tan^{-1} x &= \frac{1}{1+x^2}. \\
\frac{d}{dx} \cos^{-1} x &= -\frac{1}{\sqrt{1-x^2}}, & \frac{d}{dx} \csc^{-1} x &= -\frac{1}{|x|\sqrt{x^2-1}}, & \frac{d}{dx} \cot^{-1} x &= -\frac{1}{1+x^2}. \\
\end{align*}
\]

(2) Favorite limits:
\[
\begin{align*}
\lim_{x \to 0} \frac{\sin x}{x} &= 1, & \lim_{x \to 0} \frac{\cos x - 1}{x} &= 0, \\
\lim_{x \to 0} \frac{e^x - 1}{x} &= 1, & \lim_{x \to 0} \frac{\ln(1+x)}{x} &= 1.
\end{align*}
\]

(3) Favorite trig identities:
\[
\begin{align*}
\sin^2 x + \cos^2 x &= 1, & \sin(-x) &= -\sin x, & \cos(-x) &= \cos x, \\
\sin(xy) &= \sin x \cos y + \cos x \sin y, & \sin 2x &= 2 \sin x \cos x, & \cos(xy) &= \cos x \cos y - \sin x \sin y, & \cos 2x &= \cos^2 x - \sin^2 x.
\end{align*}
\]

(3) Favorite series:
\[
\begin{align*}
\frac{1}{1-x} &= 1+x+x^2+x^3+x^4+\ldots \\
1+x+x^2+x^3+\ldots + x^n &= \frac{x^{n+1}-1}{x-1}, \\
e^x &= 1+x+x^2/2!+x^3/3!+x^4/4!+x^5/5!+\ldots, \\
\sin x &= x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\frac{x^{11}}{11!}+\ldots, \\
\cos x &= 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\frac{x^{10}}{10!}+\ldots.
\end{align*}
\]

Things to learn FOREVER

(1) \( \frac{d}{dx} e^x = e^x, \quad e^{x+y} = e^x e^y, \quad e^0 = 1, \quad (e^a)^b = e^{ab} \)

(2) \( e^{ix} = \cos x + i\sin x \)

(3) \( \ln(ab) = \ln(a) + \ln(b), \quad \ln(a/b) = \ln(a) - \ln(b) \)

(4) Formula 1, Formula 2, Formula 3

(5) The fundamental theorem of calculus

(6) The chain rule

(7) The product rule
What is \( f(x) \) if you know its derivatives, and you can use derivatives to find series.

**Fundamental Theorem of Calculus:**
If \( f(x) \) is differentiable between \( a \) and \( b \),
\[
\int_a^b f(x) \, dx = F(b) - F(a)
\]
where \( \int f(x) \, dx = F(x) + c \),
says that
You can add up lots of little things by undoing derivatives.