Metric and Hilbert spaces

Assignment 2

Due Wednesday Oct 15 at 10am

1. Let
   \[ A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \quad \text{and} \quad B = \{(x, y) \in \mathbb{R}^2 : (x - 2)^2 + y^2 < 1\}. \]
   Determine, with proof, whether \( X = A \cup B, Y = \overline{A} \cup B \) and \( Z = \overline{A} \cup B \) are connected subsets of \( \mathbb{R}^2 \) with the usual topology.

2. Let \( X \) and \( Y \) be topological spaces and assume that \( Y \) is Hausdorff. Let \( f : X \to Y \) and \( g : X \to Y \) be continuous functions.
   (a) Show that the set \( \{x \in X \mid f(x) = g(x)\} \) is a closed subset of \( X \).
   (b) Show that if \( f : X \to \mathbb{R} \) and \( g : X \to \mathbb{R} \) are continuous then \( f - g \) is continuous.
   (c) Show that if \( f : X \to \mathbb{R} \) and \( g : X \to \mathbb{R} \) are continuous then \( \{x \in X \mid f(x) < g(x)\} \) is open.

3. Let \( X \) be a complete normed vector space over \( \mathbb{R} \). A sphere in \( X \) is a set \( S(a, r) = \{x \in X : d(x, a) = \|x - a\| = r\} \), for \( a \in X \) and \( r \in \mathbb{R}_{>0} \).
   (a) Show that each sphere in \( X \) is nowhere dense.
   (b) Show that there is no sequence of spheres \( \{S_n\} \) in \( X \) whose union is \( X \).
   (c) Give a geometric interpretation of the result in (b) when \( X = \mathbb{R}^2 \) with the Euclidean norm.
   (d) Show that the result of (b) does not hold in every complete metric space \( X \).

4. Prove that if \( X \) and \( Y \) are path connected then \( X \times Y \) is also path connected.

5. Let \( p \in \mathbb{R}_{>1} \) and define \( q \in \mathbb{R}_{>1} \) by \( \frac{1}{p} + \frac{1}{q} = 1 \).
   (a) Define the normed vector space \( \ell^p \).
   (b) Show that \( \ell^p \) is a Banach space.
   (c) Prove that the dual of \( \ell^p \) is \( \ell^q \).

6. Let \( X = C^1[0, 1] \) and \( Y = C[0, 1] \) so that functions in \( X \) are continuously differentiable and functions in \( Y \) are continuous:
   \[ Y = C[0, 1], \quad \text{with norm given by} \quad \|f\| = \sup\{|f(t)| \mid t \in [0, 1]\}, \quad \text{and} \quad X = C^1[0, 1], \quad \text{with norm given by} \quad \|f\|_0 = \|f\| + \|f'\|, \]
   where \( f' = \frac{df}{dt} \). Let \( D : X \to Y \) be the differentiation operator \( Df = \frac{df}{dt} \).
   (1) Show that \( D : (X, \|\cdot\|_0) \to (Y, \|\cdot\|) \) is a bounded linear operator with \( \|D\| = 1 \).
   (2) Show that \( D : (X, \|\cdot\|) \to (Y, \|\cdot\|) \) is an unbounded linear operator.
   (Hint: Consider the sequence of elements \( t^n \) in \( X \)).
7. Let \{a_1, a_2, \ldots \} be a bounded sequence of complex numbers. Define an operator \( T : \ell^2 \to \ell^2 \) by:

\[ T(b_1, b_2, \ldots) = (0, a_1 b_1, a_2 b_2, \ldots). \]

(1) Show that \( T \) is a bounded linear operator and find \( \|T\| \).

(2) Compute the adjoint operator \( T^* \).

(3) Show that if \( T \neq 0 \) then \( T^* T \neq TT^* \).

(4) Find the eigenvalues of \( T^* \).

8. Let \([a_{ij}]\) be an infinite complex matrix, \( i, j = 1, 2, \ldots \), such that if \( j \in \mathbb{Z}_{>0} \) then

\[ c_j = \sum_i |a_{ij}| \text{ converges, and } \quad c = \sup\{c_1, c_2, \ldots \} < \infty. \]

Show that the operator \( T : \ell^1 \to \ell^1 \) defined by

\[ T(b_1, b_2, \ldots) = \left( \sum_j a_{1j} b_j, \sum_j a_{2j} b_j, \ldots \right) \]

is a bounded linear operator and that \( \|T\| = c. \)