Assignment 2

MAST30026 Metric and Hilbert Spaces
Semester II 2016
Lecturer: Arun Ram
to be turned in before 10am on 13 October 2016

(1) (Eigenvectors)

(a) Carefully define linear operator, eigenvector and eigenvalue.

(b) Let $V$ be a complex vector space and let $T: V \to V$ be a linear operator. Prove that there exists $v \in V$ with $v \neq 0$ such that $v$ is an eigenvector of $T$.

(c) Use the proof of (b) to explicitly produce an eigenvector of the linear transformation $T: \mathbb{C}^3 \to \mathbb{C}^3$ corresponding to the matrix

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 6 & 0 & 2 \\ \pi & \sqrt{7} & 0 \end{pmatrix}. $$

(d) Let $V$ be the real vector space $\mathbb{R}^2$. Give an example of a linear transformation $T: V \to V$ that does not have a nonzero eigenvector.

(2) (Self adjoint operators) Let $T: V \to V$ be a self adjoint linear operator.

(a) Let $v$ be an eigenvector of $T$ with eigenvalue $\lambda$. Prove that $\lambda \in \mathbb{R}$.

(b) Let $\lambda$ and $\gamma$ be eigenvalues of $T$ with $\lambda \neq \gamma$. Let

$$X_\lambda = \{v \in V \mid Tv = \lambda v\} \quad \text{and} \quad X_\gamma = \{v \in V \mid Tv = \gamma v\}.$$

Prove that $X_\lambda$ is orthogonal to $X_\gamma$.

(3) (Alternative formula for the norm of a bounded self adjoint operator)

(a) Carefully define the norm of a linear operator $T: V \to V$.

(b) Carefully define bounded linear operator and self adjoint linear operator.

(c) Let $T: V \to V$ be a bounded self adjoint linear operator. Prove that

$$\|T\| = \sup\{ |\langle Tx, x \rangle| \mid \|x\| = 1 \}. $$

(4) (Existence of eigenvectors of bounded self adjoint linear operators) Let $H$ be a Hilbert space and let $T: H \to H$ be a bounded self adjoint operator.

(a) Show that there exists $x \in H$ with $\|x\| = 1$ and $|\langle Tx, x \rangle| = \|T\|$.

(b) Let $x \in H$ be as in (a). Show that $x$ is an eigenvector of $T$ with eigenvalue $\|T\|$.

(c) Use the proof of (a) to explicitly produce an eigenvector of the linear transformation $T: \mathbb{C}^3 \to \mathbb{C}^3$ corresponding to the matrix

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 5 & 0 & \pi \\ -2 & \pi & 0 \end{pmatrix}. $$

(5) (Compact linear operators) Let $H$ be a Hilbert space.

(a) Carefully define compact linear operator.

(b) Give an example (with proof) of a bounded linear operator $T: \ell^2 \to \ell^2$ which is compact and a bounded linear operator $S: \ell^2 \to \ell^2$ which is not compact.

(c) Let $T: H \to H$ be a compact linear operator. Assume $\lambda \in \mathbb{C}$ and $\lambda \neq 0$ and let

$$X_\lambda = \{v \in H \mid Tv = \lambda v\}.$$ 

Show that $X_\lambda$ is a subspace of $H$ and that $\dim(X_\lambda)$ is finite.

(6) (Expansions in orthonormal sequences) Let $H$ be a Hilbert space. Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in $H$.

(a) Let $x \in H$. Show that $\sum_{n=1}^{\infty} |\langle x, a_n \rangle|^2 \leq \|x\|^2$.

(b) Show that $P(x) = \sum_{n=1}^{\infty} \langle x, a_n \rangle a_n$, exists in $H$.

(c) Let $W = \text{span}\{a_1, a_2, \ldots\}$. With $P(x)$ as in (b), show that $P(x) \in \overline{W}$.

(d) With $W$ as in (c) and $P(x)$ as in (b), show that $x - P(x) \in \overline{W}^\perp$. 

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(7) (Fourier’s orthonormal sequences)

(a) Let \( L^2(\mathbb{T}) \) be the set of (Lebesgue measurable) functions \( f: [0, 2\pi] \to \mathbb{C} \) such that
\[
\int_0^{2\pi} |f(x)|^2 dx < \infty.
\]
Prove that \( L^2(\mathbb{T}) \) with \( \langle \cdot, \cdot \rangle : L^2(\mathbb{T}) \to L^2(\mathbb{T}) \) given by
\[
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x) \overline{g(x)} dx,
\]
is a Hilbert space.

(b) Prove that setting \( a_n = e^{inx} \) defines an orthonormal sequence \( (a_0, a_1, a_2, \ldots) \) in \( L^2(\mathbb{T}) \).

(c) Expand the function \( f(x) = x^2 \) in terms of the orthonormal sequence of (b).

(d) Evaluate the expansion in (c) at \( x = 2\pi \) to prove that
\[
\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.
\]