Lecture 21: Metric and Hilbert Spaces

**Theorem** Let $(V, ||.||)$ be a normed vector space and $d: V \times V \to \mathbb{R}_{\geq 0}$ on $V$ given by 

$$d(x,y) = ||y-x||.$$

Then $V$ is a complete metric space if and only if $V$ satisfies

$(*)$ If $(a_1, a_2, ...)$ is a sequence in $V$ and $\sum_i ||a_i||$ converges then $\sum_i a_i$ converges.

**Proof** Assume $V$ is complete.

To show: $V$ satisfies $(*)$.

Assume $(a_1, a_2, ...)$ is a sequence in $V$ and $\sum_i ||a_i||$ converges.

To show: $\sum_i a_i$ converges.

Let

$$s_n = \sum_{i=1}^{n} a_i \quad \text{and} \quad s_n = \sum_{i=1}^{n} ||a_i||$$
Since the sequence \((s_1, s_2, \ldots)\) converges, the sequence \((s_1, s_2, \ldots)\) is Cauchy.

Since \[\|s_n - s_m\| = \left\| \sum_{i=m+1}^{n} a_i \right\| \leq \sum_{i=m+1}^{n} \|a_i\| = \|s_n - s_m\|,\]

then the sequence \((s_1, s_2, \ldots)\) is Cauchy.

Since \(V\) is complete, the sequence \((s_1, s_2, \ldots)\) converges.

So \[\sum_{i=1}^{\infty} a_i\] converges.

\[\sum_{i=1}^{\infty} a_i\]

Assume that \(V\) satisfies (4).

To show: \(V\) is complete.

Let \((s_1, s_2, \ldots)\) be a Cauchy sequence in \(V\).

To show: \((s_1, s_2, \ldots)\) converges.

Using that \((s_1, s_2, \ldots)\) is Cauchy, let \(k \in \mathbb{N}\) be such that

if \(r, m \in \mathbb{N}\), then \(\|s_r - s_m\| < \frac{1}{2^n}\).
Let \( a_1 = s_{k_1}, a_2 = s_{k_2} - s_{k_1}, a_3 = s_{k_3} - s_{k_2}, \ldots \)

Then \( \|a_n\| < \frac{1}{2^n} \)

So \( \sum \|a_n\| < \sum \frac{1}{2^n} = 1 \)

So \( \sum \|a_n\| \) converges.

Since \( V \) satisfies (4) then \( \sum \|a_n\| \) converges.

So the sequence \( (s_{k_1}, s_{k_2}, \ldots) \) converges, since

\( s_{k_1} = a_1, s_{k_2} = a_1 + a_2, s_3 = a_1 + a_2 + a_3, \ldots \)

So the sequence \( (s_1, s_2, \ldots) \) has a cluster point. Since \( (s_1, s_2, \ldots) \) is Cauchy and has a cluster point then \( (s_1, s_2, \ldots) \) converges.

So \( V \) is complete.