Laboratory Class 1:  
Numbers, Sequences and Series

Aims: To explore some of the properties of rational numbers; to explore sequences in order to enhance your understanding of convergence and to illustrate the dependence of $N(\varepsilon)$ on $\varepsilon$; also to explore sequences of partial sums, which produce infinite series. In all examples, examine the MATLAB code and analyze the effect of various commands.

Before starting, copy the folder Lab1 from the lab server M&S Lab Materials\620-295 to D:\MATLAB and set the path to D:\MATLAB including subfolders.

1 The decimal expansion of rational numbers

One amazing property of rational numbers is that their decimal expansions eventually show periodicity, i.e. a repeating pattern of digits. We make this precise by the mathematical statement:

In the decimal expansion of a rational number $x = d_0.d_1d_2d_3\cdots$ there exist $n, k \in \mathbb{Z}_{>0}$ such that, for all $\ell \in \mathbb{Z}_{>0}$ with $\ell > k$, $d_{\ell+n} = d_{\ell}$

Your task is to decode this statement and use it. Are terminating expansions covered by the statement above?

Enter the following commands into the Matlab command window:

digits(100);
vpa(35/29)

What are the values of $k, n$ for the rational number $x = 35/29$?

$vpa$ stands for Variable Precision Arithmetic, which allows you to increase the number of decimal digits the computer uses, beyond the usual 16 or so. If you don’t see a period, try increasing the number of digits from 100.

Now try: $x = 35/28, 35/27, 35/26, 35/24, 35/23$ — what are $k, n$ in each case?

2 Euler’s number: code Lab1Ex2

Start with the standard sequence $a_n = (1 + \frac{1}{n})^n$.

The limit of this sequence is $e \approx 2.718281828459045\ldots$

Lab1Ex2.m is a function M-file which takes a positive integer $N$ as input computes and plots the sequence to the $N$th term. It also computes the error, that is, $|a_N - e|$ as output: type $error = \text{Lab1Ex2}(100)$ at the command line, then hit Return to run with $N = 100$. Use ↑ to recall the command and edit $N$.

Let $N(\varepsilon)$ be the minimal positive integer such that $|a_n - e| < \varepsilon$ for all $n > N(\varepsilon)$. To explore the dependence of $N(\varepsilon)$ on $\varepsilon$ consider (i) $\varepsilon = 0.1$, (ii) $\varepsilon = 0.01$, (iii) $\varepsilon = 0.001$ and (iv) $\varepsilon = 0.0001$ and determine by experiment how large $N$ needs to be to get an approximation within the prescribed error $\varepsilon$. Determine $N(\varepsilon)$ in each case:

Answers: (i) (ii) (iii) (iv)
3 Partial sums: code Lab1Ex3

A common class of sequences are formed from partial sums. Here are two examples also related to irrational numbers.

\[ a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \pm \frac{1}{n}, \quad n = 1, 2, 3, \ldots \]

\[ b_n = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \pm \frac{1}{2n-1}, \quad n = 1, 2, 3, \ldots \]

where the last term is positive or negative depending on whether \( n \) is odd or even.

Amazingly, the \( a_n \) sequence gets closer and closer to \( \log 2 \approx 0.69314718055994530941723214581765 \) and the \( b_n \) sequence gets closer and closer to \( \pi/4 \approx 0.7853981633974483096156608458198757210492923 \).

For each of the four values \( \varepsilon = 0.1, \varepsilon = 0.01, \varepsilon = 0.001 \) and \( \varepsilon = 0.0001 \) use Lab1Ex3 to experimentally determine the minimal positive integer \( N(\varepsilon) \) such that \( |a_n - \log 2| < \varepsilon \) for \( n > N(\varepsilon) \).

Answers: (i) (ii) (iii) (iv)

Repeat for \( b_n \).

Answers: (i) (ii) (iii) (iv)

Lab1Ex3.m is a function M-file which computes and plots both sequences up to \( n = N \) with \( N \) as input and both errors as output: type \([\text{error1, error2}] = \text{Lab1Ex3}(100)\) at the command line to run with \( N = 100 \).

4 Compare the partial sums: code Lab1Ex4

We explore three sequences

\[ a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}}, \quad n = 1, 2, 3, \ldots \]

\[ b_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}, \quad n = 1, 2, 3, \ldots \]

and

\[ c_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}, \quad n = 1, 2, 3, \ldots \]

One possible way to determine if the series converges to a limit or not is to compare with another series whose behaviour we know.

What do \( a_n \), \( b_n \) and \( c_n \) get closer and closer to? Run Lab1Ex4 e.g. by Lab1Ex4(100) — it demonstrates that

\[ a_n > b_n > c_n, \quad \text{for all } 1 \leq n \leq 100 \]

Prove these inequalities for all \( n > 1 \).

Later you will use inequalities like this to prove convergence or divergence of complicated series.

5 A complex series: code Lab1Ex5

Sequences and series can just as easily be constructed from complex numbers.

The geometric series familiar from school

\[ \frac{1}{1 - x} = 1 + x + x^2 + \cdots \quad \text{for } x \in \mathbb{R}, |x| < 1 \]

applies equally well for complex values:

\[ \frac{1}{1 - z} = 1 + z + z^2 + \cdots \quad \text{for } z \in \mathbb{C}, |z| < 1 \]

The code Lab1Ex5 computes and plots the partial sums of the geometric series, with the complex number \( z \) as input. Try the following and compute the modulus of \( z \) for each:

\[ > \text{Lab1Ex5}(-0.4 + \text{sqrt}(3)/2*1i) \]
\[ > \text{Lab1Ex5}(-0.6 + \text{sqrt}(3)/2*1i) \]