A topological space is a set $X$ with a specification of the open sets of $X$, i.e. a set $X$ with a collection $T$ of subsets of $X$ such that

(a) $\emptyset \in T$ and $X \in T$,

(b) if $U_i \in T$ then $\bigcup U_i \in T$,

(c) if $U \in T$, $\subset \mathbb{R}_+$ and $U_1, \ldots, U_n \in T$ then $U \cap \cdots \cap U_n \in T$.

Let $X$ and $Y$ be topological spaces.

A function $f : X \to Y$ is continuous if it satisfies

if $V$ is an open set of $Y$ then $f^{-1}(V)$ is an open set of $X$.

Let $X$ be a topological space.

A subspace of $X$ is a subset $E$ of $X$ with the topology given by making $U \cap E$ open in $E$ if $U$ is open in $X$.

**Example:** $X = \mathbb{R}$ and $E = [0,1]$.

Then $[0, \frac{1}{2})$ is not open in $X = \mathbb{R}$

but $[0, \frac{1}{2}) = (-1, \frac{1}{2}) \cap [0,1]$ is open in $[0,1]$. 
A topological space $X$ is **connected** if $X$ satisfies:

There exist open sets $A, B$ of $X$ such that

1. $A \neq \emptyset$ and $B \neq \emptyset$
2. $A \cup B = X$ and $A \cap B = \emptyset$.

Let $X$ be a topological space.

A **connected subset** of $X$ is a subset $E \subseteq X$ such that the subspace $E$ of $X$ is a connected topological space.

**Theorem** Let $f : X \rightarrow Y$ be a continuous function.

If $X$ is connected then $f(X)$ is connected.

**Proposition** Let $E \subseteq \mathbb{R}$ be a subset of $\mathbb{R}$.

$E$ is connected if and only if $E$ satisfies:

if $x, y \in E$ and $z \in \mathbb{R}$ and $x < z < y$ then $z \in E$.

**Corollary** (Intermediate Value Theorem). Let $a, b \in \mathbb{R}$ and let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. If $w \in \mathbb{R}$ and $w$ is between $f(a)$ and $f(b)$ then there exists $z \in (a, b)$ such that $f(z) = w$. 
A topological space $X$ is **Hausdorff** if $X$ satisfies:

if $x, y \in X$ and $x \neq y$ then there exist open sets $P$ and $Q$ such that $x \in P$ and $y \in Q$ and $P \cap Q = \emptyset$.

In half English: $X$ is **Hausdorff** if any two points can be separated.

![Diagram](image)

A topological space $X$ is **compact** if $X$ satisfies:

if $P \subseteq X$ and $\bigcup P = X$ then there exists a $n \in \mathbb{N}$ and $U_1, \ldots, U_n \in P$ such that $U_1 \cup U_2 \cup \ldots \cup U_n$.

In half English: $X$ is **compact** if every open cover has a finite subcover.

**Examples:**
- $[0, 1]$ is compact
- $(0, 1)$ is not compact
- $\mathbb{R}$ is not compact.
Theorem. Let $X$ be a topological space and let $E \subseteq X$.

(a) If $X$ is compact and $E$ is closed then $E$ is compact.

(b) If $X$ is Hausdorff and $E$ is compact then $E$ is closed.

(c) If $X$ is a metric space and $E$ is compact then $E$ is closed and bounded.

(d) If $X = \mathbb{R}^n$ then $E$ is compact if and only if $E$ is closed and bounded.

(e) If $X$ is a metric space then $E$ is compact if and only if $E$ satisfies:
   - if $S \subseteq E$ and $S$ is infinite then there exists $e \in E$ such that $e$ is a close point to $S$.

Part (e) in half English:

If $X$ is a metric space then $E$ is compact if and only if every infinite subset of $E$ has a close point in $E$. 