A topological space is a set $X$ with a specification of the open sets.

A **closed set** in $X$ is a subset $E$ of $X$ such that $E^c$ is open.

A **closed point** to $E$ is a point $x \in X$ such that if $N$ is a neighborhood of $x$ then $N \cap E^c \neq \emptyset$.

A subset $E$ of $X$ is **compact** if every open cover of $E$ has a finite subcover, i.e.

if $S$ is a collection of open sets of $X$ and $U \cup S = E$ then there exists $\mathcal{F} \subseteq \mathcal{S}$

such that $U, V_1, \ldots, V_n \in \mathcal{F}$ such that $U \cup V_1 \cup \cdots \cup V_n \supseteq E$.

**Theorem (a)** If $X = \mathbb{R}$ and $E \subseteq X$ then $E$ is compact if and only if $E$ is closed and bounded.

**Theorem (b)** Let $X$ be a metric space and $E \subseteq X$.

If $E$ is compact then $E$ is closed and bounded.
Let $S \subseteq X$. A cluster point of $S$ is an element $p \in X$ such that if $N$ is a neighborhood of $p$ then there exists $s \in S$ such that $s \in p$ and $s \in N$.

**Theorem** Let $X$ be a metric space and let $E \subseteq X$. Then $E$ is compact if and only if $E$ satisfies:

if $S \subseteq E$ and $S$ is infinite then there exists $e \in E$ such that $e$ is a cluster point of $S$.

**Main Theorem**

(a) let $X$ and $Y$ be topological spaces and let $f : X \to Y$ be a continuous function.

(a1) If $X$ is connected then $f(X)$ is connected.

(a2) If $X$ is compact then $f(X)$ is compact.

(b) If $f : [a,b] \to \mathbb{R}$ is continuous then

(b1) there exists $c \in [a,b]$ such that

if $x \in [a,b]$ then $f(x) \leq f(c)$.

(b2) there exists $d \in [a,b]$ such that

if $x \in [a,b]$ then $f(x) \geq f(d)$.
Mean value theorems

(a) Let $f: [a,b] \rightarrow \mathbb{R}$ be continuous and let $c \in (a,b)$ such that if $x \in [a,b]$ then $f(x) < f(c)$. If $f'(c)$ exists then $f'(c) = 0$.

(b) Same as (a) except for minimums.

(c) If $f: [a,b] \rightarrow \mathbb{R}$ is continuous and $f$ is differentiable on $(a,b)$ then there exists $c \in (a,b)$ such that $f'(c) = 0$.

(d) If $f: [a,b] \rightarrow \mathbb{R}$ is continuous and $f$ is differentiable on $(a,b)$ then there exists $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

(e) If $f: [a,b] \rightarrow \mathbb{R}$ and $g: [a,b] \rightarrow \mathbb{R}$ are continuous and $f$ and $g$ are differentiable on $(a,b)$ then there exists $c \in (a,b)$ such that

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c).$$
Example let \( f: [0, 2\pi] \rightarrow \mathbb{C} \) be given by
\[
f(x) = \cos x + i \sin x.
\]
Then \( f(0) = f(2\pi) \) but \( f'(x) \) is never 0.

Pictures

(a) \( f(x) \)

(b) \( f(c) \)
\( f(a) = f(b) \)

(c) \( f(b) \)
\( f(a) \)