Week 8 Problem Sheet
Group Theory and Linear algebra
Semester II 2011

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(1) Week 8: Vocabulary
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1. Week 8: Vocabulary

(1) Let $G$ be a group and let $H$ be a subgroup. Define a left coset of $H$, a right coset of $H$ and the index of $H$ in $G$ and give some illustrative examples.

(2) Let $G$ be a group and let $H$ be a subgroup. Define $G/H$ and give some illustrative examples.

(3) Let $G$ be a group. Define normal subgroup of $G$ and give some illustrative examples.

(4) Let $G$ be a group and let $H$ be a normal subgroup. Define the quotient group $G/H$ and give some illustrative examples.

2. Week 8: Results

(1) Let $G$ be a group and let $H$ be a subgroup of $G$. Let $a, b \in G$. Show that $Ha = Hb$ if and only if $ab^{-1} \in H$.

(2) Let $G$ be a group and let $H$ be a subgroup of $G$. Show that each element of $G$ lies in exactly one coset of $G$.

(3) Let $G$ be a group and let $H$ be a subgroup of $G$. Let $a, b \in G$. Show that the function $f : Ha \rightarrow Hb$ given by $f(ha) = hb$ is a bijection.

(4) Let $G$ be a group and let $H$ be a subgroup of $G$. Show that $G/H$ is a partition of $G$.

(5) Let $G$ be a group and let $H$ be a subgroup of $G$. Let $g \in G$. Show that $gH$ and $H$ have the same number of elements.
Let $G$ be a group of finite order and let $H$ be a subgroup of $G$. Show that $\text{Card}(H)$ divides $\text{Card}(G)$.

Let $G$ be a group of finite order and let $g \in G$. Show that the order of $g$ divides the order of $G$.

Let $G$ be a finite group and let $n = \text{Card}(G)$. Show that if $g \in G$ then $g^n = 1$.

Let $p$ be a prime positive integer. Show that if $a$ is an integer which is not a multiple of $p$ then $a^{p-1} = 1 \mod p$.

Let $p$ be a prime positive integer. Let $G$ be a group of order $p$. Show that $G$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

Let $G$ be a group and let $H$ be a subgroup of $G$. Show that $H$ is a normal subgroup of $G$ if and only if $H$ satisfies

$$\text{if } g \in G \text{ then } Hg = gH.$$ 

Let $G$ be a group and let $H$ be a normal subgroup of $G$. Show that $H$ is a normal subgroup of $G$ if and only if $H$ satisfies

$$\text{if } g \in G \text{ then } gHg^{-1} = H.$$ 

Let $G$ be a group and let $H$ be a normal subgroup of $G$. Show that if $a, b \in G$ then $HaHb = Haba$.

Let $G$ be a group and let $H$ be a normal subgroup of $G$. Show that $G/H$ with operation given by $(g_1H)(g_2H) = g_1g_2H$ is a group.

Let $f: G \to H$ be a group homomorphism. Show that $\ker f$ is a normal subgroup of $G$.

Let $f: G \to H$ be a group homomorphism. Show that $\text{im} f$ is a subgroup of $H$.

Let $f: G \to H$ be a group homomorphism. Show that $f$ is injective if and only if $\ker f = \{1\}$.

Let $G$ be a group and let $H$ be a normal subgroup of $G$. Let $f: G \to G/H$ be given by $f(g) = gH$. Show that

(a) $f$ is a group homomorphism,
(b) $\ker f = H$,
(c) $\text{im} f = G/H$.

Let $f: G \to H$ be a group homomorphism. Show that $G/\ker f \cong \text{im} f$.  

Week 8 Problem Sheet: Group Theory and Linear algebra
3. Week 8: Examples and computations

(1) Let
\[
A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]
Show that \(A\) has order 3, that \(B\) has order 4 and that \(AB\) has infinite order.

(2) Assume that \(G\) is a group such that
if \(g, h \in G\) then \((gh)^2 = g^2h^2\).
Show that \(G\) is commutative.

(3) Decide whether the positive integers is a subgroup of the integers with operation addition.

(4) Decide whether the set of permutations which fix 1 is a subgroup of \(S_n\).

(5) List all subgroups of \(\mathbb{Z}/12\mathbb{Z}\).

(6) Let \(G\) be a group, let \(H\) be a subgroup and let \(g \in G\). Show that \(gHg^{-1} = \{ghg^{-1} \mid h \in H\}\) is a subgroup of \(G\).

(7) Let \(G\) be a group and let \(g \in G\). Let \(f : G \to G\) be given by \(f(h) = ghg^{-1}\). Show that \(f\) is an isomorphism.

(8) Show that \(SO_2(\mathbb{R})\) is isomorphic to \(U_1(\mathbb{C})\).

(9) Show that \((\mathbb{R}, +)\) and \((\mathbb{R}^\times, \times)\) are not isomorphic.

(10) Show that \((\mathbb{Z}, +)\) and \((\mathbb{Q}, +)\) are not isomorphic.

(11) Show that \((\mathbb{Z}, +)\) and \((\mathbb{Q}_>0, \times)\) are not isomorphic.

(12) Show that \(SL_2(\mathbb{Z})\) is a subgroup of \(GL_2(\mathbb{R})\).

(13) Find the orders of elements 1, \(-1\), 2 and \(i\) in the group \(\mathbb{C}^\times = \mathbb{C} - \{0\}\) with operation multiplication.

(14) Find the orders of elements in \(\mathbb{Z}/6\mathbb{Z}\).

(15) Find the subgroups of \(\mathbb{Z}/6\mathbb{Z}\).

(16) Write the element (345) in \(S_5\) in diagram notation, two line notation, and as a permutation matrix, and determine its order.

(17) Write the element (13425) in \(S_5\) in diagram notation, two line notation, and as a permutation matrix, and determine its order.
Write the element (13)(24) in $S_5$ in diagram notation, two line notation, and as a permutation matrix, and determine its order.

Write the element (12)(345) in $S_5$ in diagram notation, two line notation, and as a permutation matrix, and determine its order.

Let $n$ be a positive integer. Determine if the group of complex $n$th roots of unity $\{z \in \mathbb{C} \mid z^n = 1\}$ (with operation multiplication) is a cyclic group.

Determine if the rational numbers $\mathbb{Q}$ with operation addition is a cyclic group.

Let $n$ be a positive integer. Determine if the group of complex $n$th roots of unity $\{z \in \mathbb{C} \mid z^n = 1\}$ (with operation multiplication) is a cyclic group.

Determine if the rational numbers $\mathbb{Q}$ with operation addition is a cyclic group.

Find the order of the element $(1,2)$ in the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$.

Show that the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$ and the group $\mathbb{Z}/12\mathbb{Z}$ are not isomorphic.

Show that the group $\mathbb{Z} \times \mathbb{Z}$ and the group $\mathbb{Q}$ with operation addition are not isomorphic.

Let $G$ be a group and let $a, b \in G$. Assume that $ab = ba$.

(a) Prove, by induction, that if $n \in \mathbb{Z}_{>0}$ then $ab^n = b^n a^n$.

(b) Prove, by induction, that if $n \in \mathbb{Z}_{>0}$ then $a^n b^n = b^n a^n$.

(c) Show that the order of $ab$ divides the least common multiple of the order of $a$ and the order of $b$.

(d) Show that if $a = (12)$ and $b = (13)$ then the order of $ab$ does not divide the least common multiple of the order of $a$ and the order of $b$.

Show that the order of $GL_2(\mathbb{Z}/2\mathbb{Z})$ is 6.

Let $p$ be a prime positive integer. Find the order of the group $GL_2(\mathbb{Z}/p\mathbb{Z})$.

Let $n \in \mathbb{Z}_{>0}$ and let $p$ be a prime positive integer. Find the order of the group $GL_n(\mathbb{Z}/p\mathbb{Z})$.

Show that the group $\mathbb{Z}[x]$ of polynomials with integer coefficients with operation addition is isomorphic to the group $\mathbb{Q}_{>0}$ with operation multiplication.

Let $G$ be a group with less than 100 elements which has subgroups of orders 10 and 25. Find the order of $G$.

Let $G$ be a group and let $H$ and $K$ be subgroups of $G$. Show that $|H \cap K|$ is a common divisor of $|H|$ and $|K|$.

Let $G$ be a group and let $H$ and $K$ be subgroups of $G$. Assume that $|H| = 7$ and $|K| = 29$. Show that $H \cap K = \{1\}$. 
Let $H$ be the subgroup of $G = \mathbb{Z}/6\mathbb{Z}$ generated by 3. Compute the right cosets of $H$ in $G$ and the index $|G:H|$. 

Let $H$ be the subgroup of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ generated by $(1, 0)$. Find the order of each element in $G/H$ and identify the group $G/H$. 

Let $H$ be the subgroup of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ generated by $(0, 2)$. Find the order of each element in $G/H$ and identify the group $G/H$. 

Let $n \in \mathbb{Z}_{\geq 2}$ and define $f: \text{GL}_n(\mathbb{C}) \to \text{GL}_n(\mathbb{C})$ by $f(A) = A^t$. Determine whether $f$ is a group homomorphism. 

Let $n \in \mathbb{Z}_{\geq 2}$ and define $f: \text{GL}_n(\mathbb{C}) \to \text{GL}_n(\mathbb{C})$ by $f(A) = (A^{-1})^t$. Determine whether $f$ is a group homomorphism. 

Let $n \in \mathbb{Z}_{\geq 2}$ and define $f: \text{GL}_n(\mathbb{C}) \to \text{GL}_n(\mathbb{C})$ by $f(A) = A^2$. Determine whether $f$ is a group homomorphism. 

Let $B$ be the subgroup of $\text{GL}_2(\mathbb{R})$ of upper triangular matrices and let $T$ be the subgroup of $\text{GL}_2(\mathbb{R})$ of diagonal matrices. Let $f: B \to T$ be given by

\[
f \left( \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}.
\]

Show that $f$ is a group homomorphism. Find $N = \ker f$ and identify the quotient $B/N$. 

Assume $G$ is a cyclic group and let $N$ be a subgroup of $G$. Show that $N$ is a normal subgroup of $G$ and that $G/N$ is a cyclic group. 

Simplify $3^{32} \mod 53$. 

Suppose that $2^{147052} = 76511 \mod 147053$. What can you conclude about 147053? 

Show that if $f: G \to H$ is a group homomorphism and $a_1, a_2, \ldots, a_n \in G$ then $f(a_1a_2\cdots a_n) = f(a_1)f(a_2)\cdots f(a_n)$. 

Describe all group homomorphisms $f: \mathbb{Z} \to \mathbb{Z}$. 

Show that $\text{SO}_n(\mathbb{R})$ is a normal subgroup of $\text{O}_n(\mathbb{R})$ by finding a homomorphism $f: \text{O}_n(\mathbb{R}) \to \{ \pm 1 \}$ with kernel $\text{SO}_n(\mathbb{R})$. Identify the quotient $\text{O}_n(\mathbb{R})/\text{SO}_n(\mathbb{R})$. 

Show that $\text{SU}_n(\mathbb{C})$ is a normal subgroup of $\text{U}_n(\mathbb{C})$ by finding a homomorphism $f: \text{U}_n(\mathbb{C}) \to \text{U}_1(\mathbb{C})$ with kernel $\text{SU}_n(\mathbb{C})$. Identify the quotient $\text{U}_n(\mathbb{C})/\text{SU}_n(\mathbb{C})$. 

Let $G$ be a group and let $H$ be a subgroup of $G$. Let $f: G/H \to H\backslash G$ be given by $f(aH) = Ha^{-1}$. Show that $f$ is a function and that $f$ is a bijection.
(48) Let $G = \mathbb{Z}$ and $H = 2\mathbb{Z}$. Compute the cosets of $H$ in $G$ and the index $|G:H|$.

(49) Let $G = S_3$ and let $H$ be the subgroup generated by $(123)$. Compute the cosets of $H$ in $G$ and the index $|G:H|$.

(50) Let $G = S_3$ and let $H$ be the subgroup generated by $(12)$. Compute the cosets of $H$ in $G$ and the index $|G:H|$.

(51) Let $G = \text{GL}_2(\mathbb{R})$ and let $H = \text{SL}_2(\mathbb{R})$. Compute the cosets of $H$ in $G$ and the index $|G:H|$.

(52) Let $G$ be the subgroup of $\text{GL}_2(\mathbb{R})$ given by

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \middle| x, y \in \mathbb{R}, x > 0 \right\}$$

Let $H$ be the subgroup of $G$ given by

$$H = \left\{ \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix} \middle| z \in \mathbb{R}, z > 0 \right\}.$$  

Each element of $G$ can be identified with a point $(x, y)$ of $\mathbb{R}^2$. Use this to describe the right cosets of $H$ in $G$ geometrically. Do the same for the left cosets of $H$ in $G$.

(53) Consider the set $AX = B$ of linear equations where $X$ and $B$ are column vectors, $X$ is the matrix of unknowns, and $A$ the matrix of coefficients. Let $W$ be the subspace of $\mathbb{R}^n$ which is the set of solutions of the homogeneous equations $AX = 0$. Show that the set of solutions of $AX = B$ is either empty or is a coset of $W$ in the group $\mathbb{R}^n$ (with operation addition).

(54) Let $H$ be a subgroup of index 2 in a group $G$. Show that if $a, b \in G$ and $a \notin H$ and $b \notin H$ then $ab \notin H$.

(55) Let $G$ be a group. Let $H$ be a subgroup of $G$ such that if $a, b \in G$ and $a \notin H$ and $b \notin H$ then $ab \notin H$. Show that $H$ has index 2 in $G$.

(56) Let $G$ be a group of order $841 = (29)^2$. Assume that $G$ is not cyclic. Show that if $g \in G$ then $g^{29} = 1$.

(57) Show that the subgroup $\{(1), (123), (132)\}$ of $S_3$ is a normal subgroup.

(58) Show that the subgroup $\{(1), (12)\}$ of $S_3$ is not a normal subgroup.

(59) Show that $\text{SL}_n(\mathbb{C})$ is a normal subgroup of $\text{GL}_n(\mathbb{C})$.

(60) Let $G$ be a group. Show that $\{1\}$ and $G$ are normal subgroups of $G$. 

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Show that every subgroup of an abelian group is normal.

Write down the cosets in \( \text{GL}_n(\mathbb{C})/\text{SL}_n(\mathbb{C}) \) then show that 
\[ \text{GL}_n(\mathbb{C})/\text{SL}_n(\mathbb{C}) \cong \text{GL}_1(\mathbb{C}) \, . \]

Show that the function \( \text{det}: \text{GL}_n(\mathbb{C}) \to \text{GL}_1(\mathbb{C}) \) given by taking the determinant of a matrix is a homomorphism.

Show that the function \( f: \text{GL}_1(\mathbb{C}) \to \text{GL}_1(\mathbb{R}) \) given by \( f(z) = |z| \) is a homomorphism.

Show that the determinant function \( \text{det}: \text{GL}_n(\mathbb{C}) \to \text{GL}_1(\mathbb{C}) \) is surjective and has kernel \( \text{SL}_n(\mathbb{C}) \).

Show that the homomorphism \( f: \text{GL}_1(\mathbb{C}) \to \text{GL}_1(\mathbb{R}) \) given by \( f(z) = |z| \) has image \( \mathbb{R}_{>0} \) and kernel \( U_1(\mathbb{C}) \) (the group of \( 1 \times 1 \) unitary matrices). Conclude that 
\[ \text{GL}_1(\mathbb{C})/U_1(\mathbb{C}) \cong \mathbb{R}_{>0} \, . \]

Show that the homomorphism 
\[ f: \mathbb{R} \to \text{SO}_2(\mathbb{R}) \]
\[ \theta \to \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \]
is surjective with kernel \( 2\pi \mathbb{Z} \). Conclude that 
\[ \mathbb{R}/(2\pi \mathbb{Z}) \cong \text{SO}_2(\mathbb{R}) \, . \]

Show that the set of matrices \( H = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \bigg| \ ad \neq 0 \right\} \) is a subgroup of \( \text{GL}_2(\mathbb{R}) \) and that the set of matrices \( K = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \bigg| \ b \in \mathbb{R} \right\} \) is a normal subgroup of \( H \).

Let \( G \) be a group and let \( H \) be a subgroup of \( G \). Show that \( HH = H \).

Let \( G \) be a group and let \( K \) and \( L \) be normal subgroups of \( G \). Show that \( K \cap L \) is a normal subgroup of \( G \).

Let \( G \) be a group and let \( n \) be a positive integer. Assume that \( H \) is the only subgroup of \( G \) of order \( n \). Show that \( H \) is a normal subgroup of \( G \).

Let \( G \) be an abelian group and let \( N \) be a normal subgroup of \( G \). Show that \( G/N \) is abelian.

Let \( G \) be a cyclic group and let \( N \) be a normal subgroup of \( G \). Show that \( G/N \) is cyclic.
(74) Find surjective homomorphisms from $\mathbb{Z}/8\mathbb{Z}$ to $\mathbb{Z}/8\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z}$, and $\{1\}$ (the group with one element).

(75) Let $\mathbb{R}$ denote the group of real numbers with the operation of addition and let $\mathbb{Q}$ and $\mathbb{Z}$ be the subgroups of rational numbers and integers, respectively. Show that it is possible to regard $\mathbb{Q}/\mathbb{Z}$ as a subgroup of $\mathbb{R}/\mathbb{Z}$ and show that this subgroup consists exactly of the elements of finite order in $\mathbb{R}/\mathbb{Z}$.

4. References
