(1) Define category, define the category of algebras, and prove that the category of algebras is a category.

(2) Define category, define the category of vector spaces, and prove that the category of vector spaces is a category.

(3) Define category, define the category of sets, and prove that the category of sets is a category.

(4) Define category, define the category of rings, and prove that the category of rings is a category.

(5) Define category, define the category of fields, and prove that the category of fields is a category. Show that all morphisms in the category of fields are injective.

(6) Let $A$ be an algebra. Define category, define the category of $A$-modules, and prove that the category of $A$-modules is a category.

(7) Define tensor product $V \otimes W$ of vector spaces and show that a function $f : V \times W \to Z$ is bilinear if and only if $f : V \otimes W \to Z$ is a linear transformation.

(8) Let $M$ be a finite dimensional $A$-module. Let $b_1, \ldots, b_n$ and $v_1, \ldots, v_d$ be two bases of $M$. Let

$$
\rho_b : A \to M_d(\mathbb{C}) \quad \text{and} \quad \rho_v : A \to M_d(\mathbb{C})
$$

be the corresponding algebra homomorphisms. Let $a \in A$. Determine a formula for $\rho_v(a)$ in terms of $\rho_b(a)$ and the transition matrix between the two bases.

(9) Let $f : M \to N$ be an $A$-module homomorphism. Show that $\ker f$ is a submodule of $M$ and $\text{im } f$ is a submodule of $N$.

(10) Let $d \in \mathbb{Z}_{>0}$. Show that $M_d(\mathbb{C})$ is an algebra.

(11) Show that $\mathbb{C}[x_1, \ldots, x_n]$ is an algebra.

(12) Show that $\mathbb{C}[x]$ is an algebra.
(13) Show that $\mathbb{C}$ is an $\mathbb{R}$-algebra.

(14) Let $f: A \to B$ be a morphism of algebras. Define a map

$$\text{Res}_{A}^{B}: \{B\text{-modules}\} \longrightarrow \{A\text{-modules}\} \quad \text{given by setting} \quad am = f(a)m,$$

for $m \in M$ and $a \in A$. Define functor and show that $\text{Res}_{A}^{B}$ is a functor.

(15) Let $f: A \to B$ be a morphism of algebras and let $\text{Res}_{A}^{B}$ be as defined in the previous exercise. Show that if $f$ is surjective and $M$ is a simple $B$-module then $M = \text{Res}_{A}^{B}(M)$ is a simple $A$-module.

(16) Let $A$ be an algebra and let $I$ be an ideal of $A$. Show that $A/I$ is an algebra and if $M$ is a simple $A/I$-module then $M$ is a simple $A$-module.

(17) Show that a simple module is indecomposable and give an example of an indecomposable module that is not simple.