Five "equivalences" of categories

1) Passing to the tangent space.

\[ \{ \text{groups with a tangent space} \} \longrightarrow \{ \text{Lie algebras} \} \]

\[ G \longrightarrow \mathfrak{g} = \text{Lie}(G) = T_1(G) \]

\[ \varphi: G \longrightarrow H \quad \varphi^*: \mathfrak{g} \longrightarrow \mathfrak{h} \]

where

\[ T_1(G) = \{ \text{tangent vectors to } G \text{ at } e \} \quad \delta: \mathfrak{g} \longrightarrow \mathfrak{h} \]

\[ \{ \text{one parameter subgroups of } G \} \quad \psi: \mathbb{R} \longrightarrow G \]

\[ \{ \text{left invariant vector fields on } G \} \quad \xi: C^\infty(G) \rightarrow C^\infty(G) \]

where the bijections are given by

\[ \xi(f) = (f(\varphi(t)\cdot)) \quad \text{and} \quad \delta(f) = \left. \frac{d}{dt} f(\xi(t)) \right|_{t=0} \]
If \( g : G \to H \) then \( d\psi : \mathcal{F} \to \mathcal{F} \) is given by
\[
d\psi(f) = f \circ g^*, \quad d\psi(\xi) = \xi \circ g^*, \quad d\psi(\delta) = \delta \circ g
\]
where \( g^* : \mathcal{C}^0(H) \to \mathcal{C}^0(G) \) is the morphism on rings of functions corresponding to \( g \).

\( G \) is recovered from \( \mathcal{F} \) by the exponential map \( \exp : \mathcal{F} \to G \)
\[
tX \mapsto e^{tX}
\]
where \( e^{tX} = \mathcal{s}(t) \) if \( \mathcal{s} \) is the one-parameter subgroup corresponding to \( X \).

12) "Weyl's unitary trick" and Tannaka-Krein.

\[
\{ \text{Hopf algebras} \} \leftrightarrow \{ \text{algebraic groups} \} \leftrightarrow \{ \text{compact Lie groups} \}
\]
\[
\mathcal{O}_G \longrightarrow G \longrightarrow K
\]
where \( \mathcal{O}_G \) is the ring of functions of \( G \),

\( K \) is the maximal compact subgroup of \( G \),

\( K \) is recovered from \( \mathcal{O}_G \) as
\[
K = \{ \mathfrak{g} \in \mathcal{O}_G \mid d\mathfrak{g} = g \circ g^3 \} \quad \text{the group like elements in } \mathcal{O}_G
and $\mathcal{O}_G$ is the ring of coordinate functions of finite-dimensional representations of $K$.

13) Enveloping algebras and differential operators

\[
\begin{array}{ccc}
\{ \text{Lie algebras} \} & \leftrightarrow & \{ \text{associative algebras} \} \\
\mathfrak{g} & \mapsto & U\mathfrak{g}
\end{array}
\]

where $U\mathfrak{g}$ is the enveloping algebra of $\mathfrak{g}$, the associative algebra generated by the vector space of with relations

\[xy - yx = [xy], \quad \forall x, y \in \mathfrak{g}.
\]

The functor $U$ is the left adjoint to the functor

\[
\begin{array}{ccc}
\{ \text{associative algebras} \} & \longrightarrow & \{ \text{Lie algebras} \} \\
A & \longrightarrow & \mathcal{L}(A)
\end{array}
\]

where $\mathcal{L}(A) = A$ with bracket given by

\[[a_1, a_2] = a_1 a_2 - a_2 a_1, \quad \forall a_1, a_2 \in A.
\]

$U$ is the left adjoint to $\mathcal{L}$ means

\[\text{Hom}_{\text{alg}}(U\mathfrak{g}, A) = \text{Hom}_{\text{Lie}}(\mathfrak{g}, \mathcal{L}(A))
\]

for all Lie algebras $\mathfrak{g}$ and all assoc. algebras $A$.

Remark: $U\mathfrak{g} = " \mathcal{O}_G^* "$ (the dual of $\mathcal{O}_G$).
(4) Chevalley groups

\{\text{reductive algebraic groups} \} \rightarrow \{\text{\mathbb{Z}-reflection groups}\}

G \rightarrow (\frac{\mathbb{Z}}{\mathbb{Z}}, W_0)

where, if \(T\) is a maximal torus in \(G\) then

\[ \mathbb{Z}_T = \text{Hom}(C^x, T), \quad \mathbb{Z}_{\mathbb{Z}} = \text{Hom}(T, C^x) \]

and

\[ W_0 = N(T)/T, \quad \text{where } N(T) \text{ is the normalizer of } T \text{ in } G. \]

The group \(G\) is recovered from \((\mathbb{Z}, W_0)\) as the group presented by generators

\[ x_{1/c}, x_{-1/c}, h_{-1/t}, \quad \lambda x \in \mathbb{R}^+, c, c^x \]

with relations

\[ x_{1/c}(x_{1/c} \cdot x_{1/c}) = x_{1/c} x_{1/c} x_{1/c}, \quad \text{etc.} \]

(5) \(p\)-compact groups

\{\text{\(p\)-compact groups}\} \leftrightarrow \{\text{\(\mathbb{Z}_p\)-reflection groups}\}

\[ BG \leftrightarrow (\mathbb{Z}, W_0) \]