Theorem Let $G$ be a connected linear algebraic group.
All Borel subgroups are conjugate in $G$.

Let $G$ be a connected linear algebraic group.
The set of connected closed solvable subgroups of $G$ is ordered by inclusion.

A Borel subgroup is a maximal connected closed solvable subgroup of $G$.

Example: If $G = \text{GL}_n(\mathbb{C})$ then

$$B = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \in \text{GL}_n(\mathbb{C}) \right\}$$

is a Borel subgroup of $G$.

Maximal compact subgroups

Theorem Let $G$ be a connected Lie group.
All maximal compact subgroups of $G$ are conjugate in $G$.

Let $G$ be a connected Lie group.
The set of compact subgroups of $G$ is ordered by inclusion.
Example. If $G = \mathfrak{gl}(C)$ then
\[ U_1(C) = \{ g \in \mathfrak{gl}(C) \mid g \tilde{g}^{-1} = 1 \} \]
is a maximal compact subgroup.

If $G = SU(1)$ then $SU(1) = \{ g \in SU(1) \mid g \tilde{g}^{-1} = 1 \}$
is a maximal compact subgroup.

**Maximal tori and Cartan subalgebras**

**Theorem (Boe, Lie Ch. IX §2 No 1 Theorem 1 and No 2 Theorem 2)**

Let $G$ be a compact Lie algebra.

Let $K$ be a compact Lie group.

Then maximal tori are conjugate in $G$.

The Lie algebras of the maximal tori are the Cartan subalgebras of $K = \text{Lie}(K)$.

A torus in $K$ is a subgroup $H$ isomorphic to $S^1 \times S^1 \times \cdots \times S^1$ with $r \geq 0$

The set of tori in $K$ is ordered by inclusion.

Example. $U_1(C) = \{ z \in C^x \mid z \bar{z} = 1 = e^{i \theta} \mid 0 < \theta < 2\pi \} = S^1$.

If $K = U_1(C)$ then
\[ H = \{ (e^{i \theta}, 0) \mid 0 < \theta < 2\pi \} \] is a maximal torus in $K$. 
Sylow subgroups

Theorem. Let $G$ be a finite group.

All $p$-Sylow subgroups are conjugate in $G$.

Let $p \in \mathbb{Z}_{\geq 0}$ be prime.

A $p$-group is a finite group with order a power of $p$.

Let $G$ be a finite group.

The set of $p$-subgroups of $G$ is ordered by inclusion.

A $p$-Sylow subgroup is a maximal $p$-subgroup of $G$.

Example. Let $F_p$ be the field with $p$ elements.

Then

\[
\text{Card } (\text{GL}_n(F_p)) = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})
\]

\[= q^{n^2} (q - 1)^n (q^{n-1}) \cdots (q - q^{n-1})
\]

\[= q^{n^2} (q - 1)^n (q - q^1) \cdots (q - q^{n-1})
\]

\[= \left( \sum_{w \in S_n} q^{\ell(w)} \right) (q - 1)^n q^{n+1} \cdots q^{2n-1}
\]

So, if $q = p$ then $G = GL_n(F_p)$

\[U^+ = \left\{ \begin{pmatrix} a & i \ 0 & 1 \end{pmatrix} \mid a \in \mathbb{F}_p \right\}
\]

is a $p$-Sylow subgroup of $G$. 
$G$ acts on $G/B$.

Then $B$ is the stabilizer of $B$ in $G/B$.
$G$ is certainly transitive on $G/B$.

Certainly $gBq^{-1} \leq \text{Stab}(gqB)$.

So

\{ conjugates \} $\cong$ $G/B$, since $G$ acts on \{ subgroups \} by conjugation.

In each case we get a nice homogeneous space.

$G$ acts on $G/B$ the flag variety

$K$ acts on $K/H$ the flag variety

$G$ acts on $G/K$ the upper half plane

$G$ acts on \{ $p$-Sylow \} subgroups $\cong \frac{G/H}{\text{Card}(G/K)} = 1$ mod $p$. 