$g = \text{complex reductive Lie algebra}$

$b = \text{Borel subalgebra (maximal solvable)}$

$\mathfrak{g} = \text{Cartan subalgebra (maximal abelian)}$

**Example**

$g = \mathfrak{gl}(n) = M_n(\mathbb{C})$

$b = \mathfrak{h}^0 = \{ \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \}$ and $\mathfrak{g} = \{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \}$

$g = \bigoplus \mathfrak{h} - \mathfrak{j} \bigoplus \mathfrak{j} - \mathfrak{j}$

where

$\mathfrak{h} = \{ e_i - e_j \}$

and

$[e_i - e_j, e_i - e_j] = 0$

So

$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{j} \oplus \mathfrak{r}^+$

and the simple roots are

$\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \ldots, \alpha_n = e_n - e_1$

since

$[e_i - e_j, e_j - e_k] = e_i - e_k$ for $i < j < k$. 

Example \( \mathfrak{g} = \mathfrak{s} = \mathfrak{sl}_2 \) with \( \mathfrak{h} = \mathfrak{sl}_2 \) and \( \mathfrak{g} = \mathfrak{sl}_2 \) with \( \mathfrak{h} = \mathfrak{sl}_2 \) with

\[
\mathfrak{h} = \mathfrak{sl}_2 \quad \text{with} \\
\mathfrak{h} = \mathfrak{sl}_2 \quad \text{with} \\
\mathfrak{h} = \mathfrak{sl}_2 \quad \text{with}
\]

Then the simple root is \( \alpha_1 = \epsilon_1 - \epsilon_2 \) and \( \mathfrak{g} = \mathfrak{sl}_2 \) is generated by \( \epsilon = \mathbf{e}_0 \) and \( \mathbf{f} = \mathbf{f}_0 \) with relations

\[
[\mathbf{e}, \mathbf{f}] = \mathbf{h}, \quad \text{and} \quad [\mathbf{h}, \mathbf{e}] = 2\mathbf{e} \quad \text{and} \quad [\mathbf{h}, \mathbf{f}] = 2\mathbf{f}
\]

so that \( \mathbf{h} = \mathbf{e}_0 \).

The enveloping algebra \( U \mathfrak{g} \) is the algebra given by generators \( \mathbf{e}, \mathbf{f}, \mathbf{h} \) with relations

\[
\mathbf{e} \mathbf{f} = \mathbf{f} \mathbf{e} + \mathbf{h}, \quad \mathbf{e} \mathbf{h} = \mathbf{h} \mathbf{e} - 2\mathbf{e}, \quad \mathbf{f} \mathbf{h} = \mathbf{h} \mathbf{f} + 2\mathbf{f}
\]

A \( \mathfrak{g} \)-module is a \( U \mathfrak{g} \)-module.
Let $M$ be a finite dimensional $\mathfrak{g}$-module.

$M = \oplus_{\mu \in \mathfrak{h}^*} M_\mu$ where $M_\mu = \{ m \in M \mid h_\mu = \mu(h) m \}$.

Let $\mathfrak{h} = \text{roots}$.

Let $x \in \mathfrak{h}$ and let $\alpha \in \mathfrak{h}^*$. Then, if $h \in \mathfrak{z}$ then

$h_\alpha m = ([h, \alpha] + \alpha(h)) m$

$= (h_\alpha - \alpha(h)) m$

$= (x(h) + \mu(h)) \alpha m$

$= (x + \mu) (h) \alpha m$.

So $\alpha m \in M_{x+\mu}$.

Define an order $\mathfrak{z}^*$ by $x + \mu > \mu$ for $x \in \mathfrak{h}^*$. (dominance order).

A highest weight of $M$ is $\mu \in \mathfrak{z}^*$ such that if $\alpha \in \mathfrak{z}^+$ then $\alpha M_\mu = 0$.

A highest weight vector is a vector $m \in M_\mu$ let $\mu \in \mathfrak{z}^*$. The Verma module of highest weight $\mu$ is

$M(\mu) = \bigoplus_{\lambda \in \mathfrak{h}^*} M_{\lambda}$, where
$G = \text{span}\{v^3\}$ with $ev^* = 0$ and $hv^* = \mu(h)v^*$ for $e \in \mathbb{R}^+$ and $h \in \mathbb{R}$.

Thus $M(\mu) = Uv^+$, where $U = Uy$.

Since $y = \mathbb{R} \oplus y \oplus y^+$, then $U = \mathbb{R} - U_0 U^+$

and $M(\mu) = Uv^+ = U - U_0 U^+ v^+ = U^- v^+$.

Note:
\[ U = U^- U_0 U^+ \] (where $U = Ux$, $U_0 = Uy$, $U^+ = Ux^+$)
is a version of the

Poincaré-Birkhoff-Witt Theorem:

If $\mathcal{I}$ has basis $\{d_1, d_2, \ldots\}$ then

$U\mathcal{I}$ has basis $\{d_1^m d_2^m \ldots d_n^m \mid m \in \mathbb{Z}_{\geq 0}\}$ and

$m, m_2, \ldots, m_n \in \mathbb{Z}_{\geq 0}$

So, in our case, if $\mathcal{I}$ has basis $\{f_1, \ldots, f_M\}$
$\mathcal{Y}$ has basis $\{h_{N_1}, \ldots, h_{N_2}\}$
\[ \mathcal{I}^+ \text{ has basis } \{e_1, \ldots, e_N\} \]
then $U^- \text{ has basis } \{f_{1m}^M \ldots f_{Nn}^{Mn} \mid m, \ldots, n \in \mathbb{Z}_{\geq 0}\}$
$\mathcal{Y}$ has basis $\{h_{N_1}^{K_1} \ldots h_{N_2}^{K_2} \mid k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}\}$
$U^+ \text{ has basis } \{e_{N_1}^{1m} \ldots e_{Nn}^{1n} \mid m, \ldots, n \in \mathbb{Z}_{\geq 0}\}$.