

Math127- Summary of compactness

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We divide the discussion into properties of compact sets, constructing new compact sets and using compact sets.

Definition A set S in a metric space M is compact if given any sequence of points x_n in S , there is a subsequence x_{n_i} which converges to a point x in S .

Examples

- Any closed bounded set in R^n is compact.
- Any finite set is compact
- The set $\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{k}, \dots\} \cup \{0\}$ is compact
- Any closed interval $[a, b]$ with $a \neq b$ in Q is not compact.
- Any open interval (a, b) with $a \neq b$ in R is compact.
- If $f : R^n \rightarrow R$ is a continuous function, then $f^{-1}[a, b] \cap \{x : \|x\| \leq 1\}$ is compact.

To show that such examples are compact, we need to establish key properties of compact sets and also ways of constructing compact sets.

Properties

- Any compact set must be closed and bounded.
- In R^n the converse is true - any closed and bounded set is compact (Heine Borel Theorem). But this is not true in Q .
- Any compact metric space is complete, i.e every Cauchy sequence must converge. (Reason - by compactness, a subsequence of a Cauchy sequence converges to a limit in the space. But for Cauchy sequences, it is easy to see if a subsequence converges then the whole sequence converges to the same limit).

Constructing new compact sets from old ones

- If $f : M \rightarrow P$ is continuous and U is a compact subset of M , then $f(U)$ is compact in P .
- Any closed subset of a compact set is compact.
- Any finite union of compact sets is compact

- Any intersection of compact sets is compact
- Any intersection of nested non-empty compact sets is non-empty (Nested sets U_i satisfy that $\dots U_k \subset U_{k-1} \subset \dots U_2 \subset U_1$)
- Any Cartesian product of compact sets is compact. (In this course we will only consider finite Cartesian products, but this works for infinite products as well).

As an exercise, let us consider the final example above of a compact set, namely $f^{-1}[a, b] \cap \{x : \|x\| \leq 1\}$. To see this is compact, note that since f is continuous and $[a, b]$ is closed, $f^{-1}[a, b]$ must be closed. But since we also know that $\{x : \|x\| \leq 1\}$ is closed, it follows that the intersection is an intersection of two closed sets, so is closed. But also $\{x : \|x\| \leq 1\}$ is bounded, so we see that the intersection lies inside a bounded set so is bounded. But now by Heine Borel, we see that the intersection is indeed compact.

Note that if we choose a continuous function such as $f(x, y) = x^2 - 4xy + y^3$, then the compact set we have constructed is precisely $\{(x, y) : a \leq x^2 - 4xy + y^3 \leq b, x^2 + y^2 \leq 1\}$.

Applications of compact sets

- If $f : M \rightarrow P$ is a continuous bijection between metric spaces and M is compact, then f is a homeomorphism. (This is a very useful result to cut down the work of showing maps are homeomorphisms, since we can skip the step of proving f^{-1} is continuous!)
- If $f : M \rightarrow P$ is a continuous map between metric spaces and M is compact, then f is uniformly continuous. (By uniformly continuous, we mean that given any $\epsilon > 0$, we can find a $\delta > 0$ so that for *all* $x, y \in M$, $d(x, y) < \delta$ implies $d'(f(x), f(y)) < \epsilon$. The key point here is that usually, δ depends on where x, y are in M , i.e δ depends on the choice of x say).
- If $f : M \rightarrow R$ is a real valued continuous function from a compact metric space M , then f achieves its maximum and minimum values, i.e there are points $x, y \in M$ so that $f(x) \leq f(z) \leq f(y)$ for all points $z \in M$.

Let's look at some examples to see how these applications work.

Examples

1) Assume that $f : S^2 \rightarrow R$ is a continuous function from the unit sphere in R^3 to R . How can we describe such a function first of all? Well it is easy to see that if we start with a continuous function $F : R^3 \rightarrow R$ and restrict it to $f = F|_{S^2}$, i.e $f(x, y, z) = F(x, y, z)$ so long as $\|(x, y, z)\| = 1$, then f is continuous. (It is true but beyond the scope of our course to show the converse is true, i.e every f is the restriction of an F .) There are two ways of showing this - one is to consider the image of a convergent sequence in S^2 under f . If (x_n, y_n, z_n) is a sequence in S^2 which converges to (x, y, z) , then since S^2 is closed, $(x, y, z) \in S^2$. Now as F is continuous, $F(x_n, y_n, z_n) = f(x_n, y_n, z_n)$ converges to $F(x, y, z) = f(x, y, z)$. So this shows that f is continuous. Another way of showing this is to observe that if U is open in R , then $F^{-1}(U)$ is open in R^3 , since F is continuous. But then $f^{-1}(U) = F^{-1}(U \cap S^2)$ is open in S^2 , since we have shown that open sets in S^2 are precisely intersections of open sets in R^3 with S^2 . So we see that f is continuous.

To come to the main point of this example, given such a continuous function $f : S^2 \rightarrow R$, f must take its maximum and minimum values on S^2 , i.e there are points (x_0, y_0, z_0) and (x_1, y_1, z_1) on S^2 so that $f(x_0, y_0, z_0) \leq f(x, y, z) \leq f(x_1, y_1, z_1)$ for all $(x, y, z) \in S^2$.

On the other hand, if we instead take a continuous map $g : B^3 \rightarrow R$, where B^3 is the open unit ball, i.e $\{(x, y, z) : x^2 + y^2 + z^2 < 1\}$, then g may not achieve its maximum nor its minimum values. The simplest possible g is $g(x, y, z) = x$ and this takes values precisely in $(-1, 1)$ so there is no maximum nor minimum achieved. Note that B^3 is not compact!

2) The function $\frac{1}{x} : (0, 1] \rightarrow R$ is continuous but not uniformly continuous and again $(0, 1]$ is not compact. The reason is that if we choose $x = \frac{1}{n}$ and try to find δ so that $|x - y| < \delta$ implies $|\frac{1}{x} - \frac{1}{y}| < \epsilon$, then we have to solve $|\frac{1}{n} - y| < \delta$ and $|n - \frac{1}{y}| < \epsilon$. The second inequality is just $\frac{1}{n+\epsilon} < y < \frac{1}{n-\epsilon}$. The first inequality is $-\delta + \frac{1}{n} < y < \delta + \frac{1}{n}$. In particular, the *width* of the first interval must be no larger than the second. But the first has width 2δ and the second $\frac{1}{n-\epsilon} - \frac{1}{n+\epsilon}$ which has 'first order term' $\frac{2\epsilon}{n^2}$, expanding as geometric series. So we see that δ has to have size roughly $\frac{2\epsilon}{n^2}$ and this cannot be independent of n .