

Math 127 A 2

Winter quarter, 2005

Notes on connectedness

Definition

A space M is connected if it cannot be written as a disjoint union of non-empty open subsets. So there is no decomposition $M = A \cup B$ with A, B open in M and $A, B \neq \emptyset$.

Notice that if M is *disconnected*, i.e. not connected, then the open sets A, B will be closed as well, since the complement of A is B and vice versa. So A, B are *clopen* sets, i.e. both open and closed.

Example

We give an example showing that care must be taken when dealing with subspaces rather than subspaces. Let N be the natural numbers $\{1, 2, \dots\}$. We define a topology on N by decreeing that a subset U of N is open if and only if either $N \setminus U$ is finite or U is empty. It is easy to show this is a topology, since a finite intersection of such open sets is open and an arbitrary union of such open sets is open. It is then easy to see that N is connected, for any two non-empty open sets must intersect (in an infinite set!). BUT, if we take the subset $\{1, 2\}$ of N this is disconnected. The reason is that $\{1\} = A$ and $\{2\} = B$ are disjoint non-empty open subsets whose union is $\{1, 2\}$. The reason is that $A = U \cap \{1, 2\}$, $B = V \cap \{1, 2\}$, where $U = N \setminus \{2\}$ and $V = N \setminus \{1\}$. So since U, V are open in N , A, B are open in the subspace $\{1, 2\}$. Note here that the open sets in the subspace are disjoint but the corresponding open sets in the whole space do meet! This example shows the difference between open sets in N and in $\{1, 2\}$.

We start with five basic ways of constructing connected spaces. Note that a path in a space M is just a continuous map $f : [a, b] \rightarrow M$, where $a < b \in \mathbb{R}$. A space M is called path connected if given any two points $x, y \in M$ there is a path f with $f(a) = x$ and $f(b) = y$.

- If $f : M \rightarrow P$ is a continuous map and M is connected then $f(M)$ is connected. More generally, if U is a connected subset of M then so is $f(U)$.
- If A_i are connected subsets of a space M , which all contain a point x , then $\cup_i A_i$ is connected.

- if M is a path connected space, then M is connected.
- If S is connected, then any subset T with $S \subset T \subset \bar{S}$ is connected, where \bar{S} is the closure of S . In particular this means the closure of a connected set is connected.
- The Cartesian product of finitely many connected sets is connected.

Next, we give some standard examples of connected sets and spaces.

- R^n is connected for any n . Similarly intervals in R of the form $[a, b]$ or (a, b) or $[a, b)$ are all connected.
- Any convex set in R^n is connected - the easiest way of seeing this is to use the fact that a convex set is path connected.
- The topologists sine curve is connected, but not path connected. So this is an important example showing that path connected is stronger than connected. Recall this is the set of points in R^2 of the form $\{(x, y) : x = 0, -1 \leq y \leq 1; y = \sin \frac{1}{x}, \text{ for } x > 0\}$. To see this is connected, note that $\{(x, y) : y = \sin \frac{1}{x}, \text{ for } x > 0\}$ is the image of a continuous map of R into R^2 so is connected. (Any graph of a real function defined on R will be connected). But then the topologists sine curve is exactly the closure of this connected graph and so is connected.

Finally we give some applications of connected sets.

- The general intermediate value theorem. This states that if $f : M \rightarrow R$ is continuous and M is connected, if $a < b$ are values of f , i.e there are points $x, y \in M$ with $a = f(x), b = f(y)$, then for any $a < c < b$, there is a point $z \in M$ with $f(z) = c$.
- Connectivity is a great way of showing that certain spaces cannot be homeomorphic. For if M, P are homeomorphic via some map $f : M \rightarrow P$, then given any subset X of M , $f : M \setminus X \rightarrow P \setminus f(X)$ will also be a homeomorphism. But if $M \setminus X$ is connected and $P \setminus f(X)$ is disconnected, or the other way around, then this is impossible. So for example, we can show that R and R^2 are not homeomorphic, taking X to be a single point in R . It is easy to see that $R \setminus X$ is not connected, but $R^2 \setminus f(X)$ is path connected and so connected.
- Finally we give the idea of components (‘pieces’) of a space. Given $x \in M$, we can define C_x to be the union of all connected subsets of M containing x . We know from above that C_x will be connected - it is clearly the maximal connected subset of M containing x . Hence also

C_x will be closed, since the closure of a connected set is connected. Finally we can define an equivalence relation on the points of M by $x \sim y$ if and only if $C_x = C_y$. We call C_x a *connected component* of M .

Example

Let $M = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$. Then the connected component of each point $x \in M$ is easily seen to be just $\{x\}$. But also note although each one point subset is closed in M , the subset $\{0\}$ is *not* open in M . So it is *not* true that connected components need to be open.