Math 127 A 2

Winter quarter, 2005

Cuts versus Cauchy sequences

We list the important properties of cuts and Cauchy sequences and compare these.

A cut is a subset A of the rational numbers $\mathbb{Q}$ with the following properties;

1. $A \neq \mathbb{Q}$ and $A \neq \emptyset$
2. If $r \in \mathbb{Q}$ and $s \in A$ with $r < s$ then $r \in A$
3. A has no largest element.

A rational cut $r^* = \{s \in \mathbb{Q} : s < r\}$. So the two types of cuts are rational ones and irrational ones, of the form $A' = \{r \in \mathbb{Q} : r^2 < 2\}$. The cut A' represents $\sqrt{2}$. We can view cuts as 'representing' real numbers, i.e we can associate a cut with each real number. Two very convenient properties of real numbers which come out of this representation is the usual ordering and completeness.

We define $x \leq y$ for real numbers x, y represented by cuts A, A' if $A \subset A'$. (Here $\subset$ means that one set is contained in another and they might coincide).

Then for example, the standard properties of order of the real numbers follow from this definition, namely;

1. Given real numbers x, y, either $x < y$ or $x = y$ or $x > y$.
2. If $x < y$ and $y < z$ then $x < z$.
3. $x < y$ implies $x + z < y + z$

Completeness (the absence of any 'gaps' in the real numbers) is given by the least upper bound property, or lub for short. Namely an upper bound b for a non empty set S of real numbers satisfies $r \leq b : r \in S$. So assuming that S has an upper bound, we can consider the set $\mathbb{B}$ of all upper bounds of S. Then the lub of $\mathbb{B}$ is just a smallest element, hence a smallest or least upper bound.

Examples

1. For a cut A, the lub of A is exactly the real number x associated with the cut. So $\sqrt{2}$ is the lub of the cut $A = \{r \in \mathbb{Q} : r^2 < 2\}$. For an upper bound of A is clearly a real number at least as big as $\sqrt{2}$ so $\sqrt{2}$ is the smallest upper bound, i.e the lub.

2. For a set S which is an increasing convergent sequence, e.g $S = \{0, 1 - \frac{1}{2}, 1 - \frac{1}{3}, \ldots, 1 - \frac{1}{n}, \ldots\}$ it is easy to see that the lub is just the limit 1 of the sequence. So we can see that limits and lubs are closely connected.

The fundamental completeness property of the reals is the lub property, namely

Theorem

If S is a non-empty set of real numbers which has an upper bound, then S has a lub.

The proof of this is an easy argument about unions of cuts being a cut. Namely one just takes all the cuts corresponding the real numbers in S and considers their union. The result is then that this is a cut and also that it is indeed the lub.

The bad news about cuts is that it is very tedious proving all the field properties of the real numbers. We list them;

1. For $x, y \in \mathbb{R}$, $x + y = y + x$

2. For $x, y, z \in \mathbb{R}$, $(x + y) + z = x + (y + z)$

3. There is a real number 0 satisfying $x + 0 = 0 + x = x$, for all $x \in \mathbb{R}$

4. For any $x \in \mathbb{R}$, there is a real number $-x$ satisfying $x + (-x) = (-x) + x = 0$

5. For $x, y \in \mathbb{R}$, $x \times y = y \times x$

6. For $x, y, z \in \mathbb{R}$, $(x \times y) \times z = x \times (y \times z)$

7. There is a real number 1 so that $x \times 1 = 1 \times x = x$, for all $x \in \mathbb{R}$.

8. For any $x \in \mathbb{R}$ with $x \neq 0$, there is a real number $\frac{1}{x}$ satisfying $x \times \frac{1}{x} = \frac{1}{x} \times x = 1$

9. For $x, y, z \in \mathbb{R}$, $x \times (y + z) = x \times y + x \times z$

Checking all these laws using cuts is a very long piece of work, done in Landau's little book, Foundations of Analysis.

The notion of a Cauchy sequence can also be used to construct the real numbers, starting at the rational numbers. We will discuss this towards the end of the course. The advantage is that we don't use the order property of the real numbers, so many other spaces can be 'completed' using the same construction.

A Cauchy sequence is a sequence of real numbers $\{a_n\} = \{a_1, a_2, \dots, a_n, \dots\}$ having the following property; given any (small) positive real number $\epsilon > 0$, we can find a (large) integer N, so that if both $i, j > N$, then $|a_i - a_j| < \epsilon$.

The idea is to define an equivalence relation on Cauchy sequences of rational numbers. We want to capture the idea of two Cauchy sequences having the same real limit point. So a real number will be defined as the equivalence class of Cauchy sequences, all having the same limit. To do this, define two Cauchy sequences $\{a_n\}$, $\{b_n\}$to be equivalent (sometimes called coCauchy) if their 'union' is a Cauchy sequence. By union, we mean the new sequence $\{c_n\}$ given by $c_{2k-1} = a_k, c_{2k} = b_k$. So the new sequence is just $a_1, b_1, a_2, b_2, \dots$. It is now a (not difficult) exercise to show that demanding that c_n is Cauchy does indeed define an equivalence relation on the set of Cauchy sequences. It is also true that c_n is Cauchy if and only if a_n and b_n have the same limit, once we know all the properties of the real numbers.

For convenience, we will call the subset $\{a_{N+1}, a_{N+2}, \dots\}$ a tail of the sequence. So the Cauchy property says that the tail is contained in a small interval, in fact of width at most 2ϵ. The reason is that we can pick $j = N + 1$ in the definition of Cauchy. Then we see it gives that for all $i > N$, $|a_i - a_{N+1}| < \epsilon$. Hence the whole tail is contained in the interval $(a_{N+1} - \epsilon, a_{N+1} + \epsilon)$.

On the other hand, recall that a sequence $\{a_n\} = \{a_1, a_2, \dots, a_n, \dots\}$ is convergent with limit L if given any positive ϵ, there is an integer N so that for all $i > N$, $|a_i - L| < \epsilon$.

So we can test for convergence of sequences *without knowing their limits* by using the fundamental Cauchy convergence criterion.

Theorem

A sequence $\{a_n\} = \{a_1, a_2, \dots, a_n, \dots\}$ is convergent if and only if it is Cauchy.

Some good exercises to understand Cauchy sequences are the following, basically from Abbott, Understanding analysis, Springer 2001

Give an example of each of the following or explain why no such example can be found.

1. A Cauchy sequence which is monotonically increasing or decreasing.
2. A Cauchy sequence which is not monotonically increasing or decreasing.

3. A monotonically increasing or decreasing sequence which is not Cauchy.
4. A Cauchy sequence with a divergent subsequence.
5. An unbounded sequence with a subsequence which is Cauchy.

Suppose that $\{a_n\}$ and $\{b_n\}$ are both Cauchy sequences.

1. Show that $c_n = |a_n - b_n|$ is also Cauchy.
2. Show that $d_n = a_n + b_n$ is Cauchy.
3. (More difficult) Prove that $a_n \times b_n$ is Cauchy.
4. Would $\frac{a_n}{b_n}$ always be Cauchy? Try to find a condition which describes when this is true. (Coming up with an answer in all cases is not easy!).

Examples

1. The sequence $\{1, -1, 1, -1, \dots\}$ is not Cauchy.
2. The sequence $\{1, 2, 3, 4, \dots\}$ is not Cauchy
3. The sequence $\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \dots\}$ is Cauchy.
4. The sequence $\{1, 1, 1, 1, \dots\}$ is Cauchy.