Math 127 A 2

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Notes on equivalence classes

A *relation* is defined on a set $\mathcal{S}$ by choosing a *subset* $\mathcal{R}$ of the set of all ordered pairs of elements $\mathcal{S} \times \mathcal{S}$. For convenience, we usually write $x \sim y$ and read this as x is *related* to y, rather than saying that the pair (x, y) is in the relation subset $\mathcal{R}$.

The following three axioms then give an *equivalence relation.*

1. *reflexive*, $x \sim x$ for all elements $x \in S$.
2. *symmetric*, if $x \sim y$ then $y \sim x$
3. *transitive* if $x \sim y$and $y \sim z$, then $x \sim z$.

(Note we could also write this in terms of $\mathcal{R}$ by saying instead that $(x, x) \in \mathcal{R}$ for all $x \in S$, if $(x, y) \in \mathcal{R}$ then $(y, x) \in \mathcal{R}$, and finally, if $(x, y) \in \mathcal{R}$ and $(y, z) \in \mathcal{R}$ then $(x, z) \in \mathcal{R}$).

The *equivalence class* of an element $x \in S$ is $\{y \in S : x \sim y\}$.

A *partition* of a set $\mathcal{S}$ is a collection of subsets $\{\mathcal{P}_i : i \in \mathcal{I}\}$ with the properties that

1. $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$
2. $\bigcup\{\mathcal{P}_i : i \in \mathcal{I}\} = \mathcal{P}$

The fundamental theorem for equivalence relations is then the following:

Theorem

1. The equivalence classes of an equivalence relation either coincide or are disjoint. They form a partition of $\mathcal{S}$
2. A partition $\{\mathcal{P}_i : i \in \mathcal{I}\}$ gives an equivalence relation, by defining $x \sim y$ if and only if x, y are in the same set $\mathcal{P}_i$ of the partition.

Examples

1. Let $\mathcal{S} = \{A, B, C, D, E, F\}$ and let $\mathcal{P}_1 = \{A, B, C\}$, $\mathcal{P}_2 = \{D, E\}$ and $\mathcal{P}_3 = \{F\}$. Then the equivalence class of A is $[A] = \mathcal{P}_1 = [B] = [C]$ and similarly $[D] = [E] = \mathcal{P}_2$, with $[F] = \mathcal{P}_3$.

2. Let $\mathcal{S} = \mathbb{Z}$. Let the relation $\sim$ be defined by $m \sim n$ if and only if $m - n = 5k, k \in \mathbb{Z}$, i.e the difference between m and n is divisible by 5. Then the equivalence classes are given by

   $... = [-5] = [0] = [5] = ... = \{..., -5, 0, 5, 10, 15, ...\}$

   $... = [-4] = [1] = [6] = ... = \{..., -4, 1, 6, 11, 16, , ...\}$

   $... = [-3] = [2] = [7] = ... = \{..., -3, 2, 7, 12, 17, ...\}$

   $... = [-2] = [3] = [8] = ... = \{..., -2, 3, 8, 13, 18, ...\}$

   $... = [-1] = [4] = [9] = ... = \{..., -1, 4, 9, 14, 19, ...\}$

   Note that we get five equivalence classes $\{[0], [1], [2], [3,], [4]\}$ which are exactly the *remainders* when an integer is divided by 5. These five classes form a nice algebraic object called a *ring* where we can add, subtract and multiply remainders, noting that $[2] \times [3] = [6] = [1]$ etc. This is called modulo five arithmetic.

3. We could define an equivalence relation on the set of houses by saying that two houses are related if they are painted the same color. So all houses which are white would form an equivalence class, etc. (There is some ambiguity here, since colors can have shades, e.g grey!).

4. A non example is given by the relationship of first cousins. First cousins are defined as people having at least one common grandparent. If A is a first cousin of B and B is a first cousin of C, then it is easy to see that A need not be a first cousin of C. The first two properties (reflexive and symmetric) are satisfied, with the convention that A is a first cousin of A.

5. The relationship of brother/sister, often called siblings, is an equivalence relation. Here we define siblings as having the same parents. So it is easy to see that this is reflexive, symmetric and transitive.