1. Suppose that $A$ and $B$ are cuts of rational numbers.
We set $A + B = \{ r + s : r \in A \text{ and } s \in B \}$. We will show
that $A + B$ is a cut.

i) Show that $A + B \neq \emptyset$ and $A + B \neq \mathbb{Q}$:
Since $A \neq \emptyset$ and $B \neq \emptyset$, $A + B$ is clearly nonempty.
To show $A + B \neq \mathbb{Q}$, we note that since $A$ and $B$
are cuts, $\exists a \in A \setminus \mathbb{Q} \setminus A$ and $\exists b \in B\setminus A$. Moreover, $a > b$ for every $a \in A$; $\beta > b$ for every $b \in B$. Hence $\alpha + \beta > a + b$ for every pair $\alpha \in A$ and $\beta \in B$. Thus $\alpha + \beta \notin A + B$. Therefore $A + B \neq \mathbb{Q}$.

ii) Show that given any $c \in A + B$, $\{ r \in \mathbb{Q} : r < c \} \subseteq A + B$:
Given $c \in A + B$, $\exists a \in A$ and $\beta \in B$ s.t. $c = a + \beta$. Suppose $r \in \mathbb{Q}$ with $r < c$. Hence $r - a < b$. Since $B$ is a cut, $r - a \in B$. Thus $r = a + (r - a) \in A + B$, as desired.

iii) Show that $A + B$ has no largest element:
Let $c \in A + B$ be arbitrary. We show that $\exists \gamma \in A + B$ s.t.
$c < \gamma$. Let's write $c = a + \beta$ for some $a \in A$ and $\beta \in B$.
Since $A$ and $B$ are cuts, $\exists \alpha \in A$ and $\beta' \in B$ s.t.
$\alpha > a$ and $\beta' < \beta$. Hence $\alpha + \beta > a + \beta' = c$. We
now set $\gamma = \alpha + \beta$. Then $\gamma \in A + B$ and $c < \gamma$.

Remark: in iii), we could have set $\gamma = b$. 


2. Suppose \( A \) and \( B \) are cuts of rational numbers. Consider the set \( A-B \) defined as \( \{ r-s : r \in A \text{ and } s \in B \} \). We explain why \( A-B \) is not a cut.

It suffices to show that \( A-B \) is not bounded above.* Let \( r \in A \) and \( s \in B \) be fixed elements. Since \( B \) is a cut, \( \{ s \in \mathbb{Q} : s < s_0 \} \subseteq B \). Whence

\[
\{ r-s : s < s_0 \} \subseteq A-B.
\]

Let \( s_n \) be a sequence of elements of \( B \) such that \( \lim s_n = -\infty \). Then \( \lim r_o - s = -\infty \). This shows that \( A-B \) is unbounded from above. Therefore \( A-B \) is not a cut.

*Note that a cut always has an upper bound in \( \mathbb{Q} \). If \( C \) is a cut, then \( C \neq \mathbb{Q} \). Then \( \exists x \in \mathbb{Q} \cap C \). Whence \( x < \infty \). Then, by property (iii) in the definition of a cut.

Remark: It is amusing to note that if \( A-B = 0 \) \( = \{ r \in \mathbb{Q} : r > 0 \} \), then \( A-B = \mathbb{Q} \).

For, let \( r \in \mathbb{Q} \). If \( r = 0 \), write \( r = -1 + (-1) \). If \( r > 0 \), write \( r = 2r - r \). If \( r < 0 \), write \( r = (-r) + (-2r) \). We see that \( A-B \), in this case, contains \( \mathbb{Q} \).
3. We show that \( \sqrt{6} \) and \( \sqrt{2} + \sqrt{3} \) are both irrational.

To show that \( \sqrt{6} \in \mathbb{Q} \), prove by contradiction. If \( \sqrt{6} \) were rational, we could write \( \sqrt{6} = \frac{m}{n} \) where \( m, n \in \mathbb{N} \).

By squaring both sides of this equation and multiplication by \( n^2 \), we see that \( 6n^2 = m^2 \). Now consider prime factorizations of \( 6n^2 \) and \( m^2 \). We consider zero as an even number here. So \( 6n^2 \) has an odd number of factors of 2, while \( m^2 \) has an even number of factors of 2. Thus \( 6n^2 \neq m^2 \), a contradiction.

To show that \( (\sqrt{2} + \sqrt{3}) \in \mathbb{Q} \), we use the previous result.

Set \( x = \sqrt{2} + \sqrt{3} \). Then \( x^2 = 2 + 2\sqrt{6} + 3 \Rightarrow \frac{x^2 - 5}{2} = \sqrt{6} \).

We note that \( \mathbb{Q} \) is field, so arithmetic is closed in \( \mathbb{Q} \). So, \( x \in \mathbb{Q} \Rightarrow \frac{x^2 - 5}{2} \in \mathbb{Q} \). Hence \( \sqrt{6} \in \mathbb{Q} \), a contradiction.
4. We consider sequences of the form \( \{s_1, s_2, \ldots \} \) and determine which ones are Cauchy.

We know that the Cauchy criterion for \( \mathbb{R} \) holds (i.e. \( \mathbb{R} \) is a complete metric space): \( (s_n) \) is Cauchy \( \iff (s_n) \) converges in \( \mathbb{R} \).

- \( \{1, 1 \frac{1}{2}, 2, 2 \frac{1}{2}, 3, 3 \frac{1}{2}, \ldots \} \) is not Cauchy:
  \[
  \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( \frac{n+1}{2} \right) = \infty, \quad \text{so} \quad (s_n) \text{ diverges} \iff (s_n) \text{ is not Cauchy}.
  \]
  Directly, set \( \varepsilon = 1/3 \). Given \( N \in \mathbb{N} \), set \( n = N+1 \) and \( m = N+2 \). Then \( n, m > N \), yet
  \[
  |s_n - s_m| = \left| \frac{N+1}{2} - \frac{N+2}{2} \right| = \frac{1}{2} > \varepsilon.
  \]

- \( \{2, 2 \frac{1}{4}, 2 \frac{1}{8}, 2 \frac{1}{16}, \ldots \} \) is Cauchy:
  \[
  \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( 2 + \frac{1}{2^n} \right) = 2, \quad \text{so} \quad (s_n) \text{ is Cauchy}.
  \]
  Directly, let \( \varepsilon > 0 \). Then \( \exists N \in \mathbb{N} \) s.t. \( \forall N < \varepsilon/2 \Rightarrow 2/N < \varepsilon \).
  For any \( n, m > N \), we have
  \[
  |s_n - s_m| = \left| \frac{1}{n^2} - \frac{1}{m^2} \right| < \frac{1}{n^2} + \frac{1}{m^2} < 2/N < \varepsilon
  \]

- \( \{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots \} \) is Cauchy:
  \[
  \lim_{n \to \infty} (-1)^n/n = 0, \quad \text{so} \quad (s_n) \text{ is Cauchy}.
  \]
  Directly, let \( \varepsilon > 0 \). Then \( \exists N \in \mathbb{N} \) s.t. \( \forall N < \varepsilon/2 \Rightarrow 2/N < \varepsilon \).
  For any \( n, m > N \), \( |s_n - s_m| = |\frac{1}{n} + \frac{1}{m}| < \frac{1}{n} + \frac{1}{m} < 2/N < \varepsilon \).

- \( \{1, -1 \frac{1}{2}, 1 \frac{1}{3}, -1 \frac{1}{4}, \ldots \} \) is not Cauchy:
  Set \( m_k = 2k \) and \( n_k = 2k+1 \) for \( k = 1, 2, 3, \ldots \)
  Then \( \lim_{k \to \infty} s_{m_k} = -1 \) while \( \lim_{k \to \infty} s_{n_k} = 1 \). So \( (s_n) \) has two subsequential limits. Thus \( (s_n) \does not converge \iff (s_n) \text{ is not Cauchy} \).
  Directly, set \( \varepsilon = 1 \). Given \( N \in \mathbb{N} \), set \( n = N+1 \) and \( m = N+2 \).
  Then \( n \text{ is odd} \iff m \text{ is even} \). Hence
  \[
  |s_n - s_m| = \left|(-1)^{m+2} \left( 1 + \frac{1}{N+1} \right) - (-1)^{n+2} \left( 1 + \frac{1}{N+2} \right) \right| = \left| \left( 1 + \frac{1}{N+1} \right) + \left( 1 + \frac{1}{N+2} \right) \right| > 1 = \varepsilon.
  \]
5. We consider two functions \( \mathbb{R}^2 \to \mathbb{R} \), and determine which ones are inner products.

- \( \langle (a, b), (x, y) \rangle = ax - by \) is not an inner product.
  In particular, \( \langle (0, 1), (0, 1) \rangle = -1 \) which violates positivity.
  In general, \( \langle (a, b), (a, b) \rangle = a^2 - b^2 < 0 \) whenever \(|a| < |b|\).

- \( \langle (a, b), (x, y) \rangle = 2ax + 3by \) is an inner product.
  We check the conditions:

  (Symmetry) \( \langle (a, b), (x, y) \rangle = \langle (x, y), (a, b) \rangle \) is clear since \( \mathbb{R} \) is commutative.

  (Additivity) \( \langle (a, b), (x, y) + (c, d) \rangle = 2a(x+c) + 3b(y+d) \)
  \( = (2ax + 2ac) + (3by + 3bd) \)
  \( = (2ax + 3by) + (2ac + 3bd) \)
  \( = \langle (a, b), (x, y) \rangle + \langle (a, b), (c, d) \rangle \)

  (Scares) \( \lambda \langle (a, b), (x, y) \rangle = \lambda (2ax + 3by) \)
  \( = 2\lambda a x + 3\lambda b y \)
  \( = 2(\lambda a) x + 3(\lambda b) y \)
  \( = \langle (\lambda a, \lambda b), (x, y) \rangle \)
  \( = \langle \lambda (a, b), (x, y) \rangle \)

  (Positivity) \( \langle (a, b), (a, b) \rangle = 2a^2 + 3b^2 > 0 \) is clear.
  It is also clear that \( 2a^2 + 3b^2 = 0 \) if and only if \( a = b = 0 \).
(a. We use question 5 to prove the inequality

\[(2ax + 3by)^2 \leq (2a^2 + 3b^2)(2x^2 + 3y^2).\]

Since \[\langle (a, b), (x, y) \rangle = 2ax + 3by\] defines an inner product on \(\mathbb{R}^2\) (shown in previous problem), we may use the norm \(\|(a, b)\| = \sqrt{2a^2 + 3b^2}\).

The Cauchy-Schwartz inequality yields

\[(2ax + 3by)^2 = |\langle (a, b), (x, y) \rangle|^2 \leq \|(a, b)\|^2 \cdot \|(x, y)\|^2\]

\[= (2a^2 + 3b^2)(2x^2 + 3y^2).\]