Math 127A - 2 Midterm 1 Solutions

1. Suppose that A and B are cuts of rational numbers. We set $A+B = \{r+s : r \in A \text{ and } s \in B\}$. We will show that $A+B$ is a cut.

i) Show that $A+B \neq \emptyset$ and $A+B \neq \mathbb{Q}$:
Since $A \neq \emptyset$ and $B \neq \emptyset$, $A+B$ is clearly nonempty.
To show $A+B \neq \mathbb{Q}$, we note that since A and B are cuts, $\exists\, \alpha \in \mathbb{Q} \setminus A$ and $\beta \in \mathbb{Q} \setminus B$. Moreover, $\alpha > a$ for every $a \in A$; $\beta > b$ for every $b \in B$. Whence $\alpha + \beta > a + b$ for every pair $a \in A$ and $b \in B$. Thus $\alpha + \beta \notin A+B$. Therefore $A+B \neq \mathbb{Q}$.

ii) Show that given any $c \in A+B$, $\{r \in \mathbb{Q} : r < c\} \subseteq A+B$:
Given $c \in A+B$, $\exists\, a \in A$ and $b \in B$ s.t. $c = a+b$. Suppose $r \in \mathbb{Q}$ with $r < c$. Hence $r - a < b$. Since B is a cut, $r-a \in B$. Thus $r = a + (r-a) \in A+B$, as desired.

iii) Show that $A+B$ has no largest element:
Let $c \in A+B$ be arbitrary. We show that $\exists\, \gamma \in A+B$ s.t. $c < \gamma$. Let's write $c = a+b$ for some $a \in A$ and $b \in B$. Since A and B are cuts, $\exists\, \alpha \in A$ and $\beta \in B$ such that $\alpha > a$ and $\beta > b$. Whence $\alpha + \beta > a + b = c$. We now set $\gamma = \alpha + \beta$. Then $\gamma \in A+B$ and $\gamma > c$.

<u>Remark:</u> in iii), we could have set $\beta = b$.

2. Suppose A and B are cuts of rational numbers. Consider the set $A-B$ defined as $\{r-s : r\in A \text{ and } s\in B\}$. We explain why $A-B$ is not a cut.

It suffices to show that $A-B$ is not bounded above.* Let $r_0\in A$ and $s_0\in B$ be fixed elements. Since B is a cut $\{s\in\mathbb{Q} : s<s_0\}\subseteq B$. Whence

$$\{r_0-s : s<s_0\}\subseteq A-B.$$

Let s_n be a sequence of elements of B such that $\lim_{n\to\infty} s_n=-\infty$. Then $\lim_{n\to\infty} r_0-s=\infty$. This shows that $A-B$ is unbounded from above. Therefore $A-B$ is not a cut.

*Note that a cut always an upper bound in $\mathbb{Q}$. If C is a cut, then $C\neq\mathbb{Q}$. Then $\exists\, \gamma\in\mathbb{Q}\setminus C$. Whence $\gamma > c_0$ $\forall c_0\in C$, by property (ii) in the definition of a cut.

<u>Remark</u>: It is amusing to note that if $A=B=0^*=\{r\in\mathbb{Q} : r<0\}$, then $A-B=\mathbb{Q}$:

For, let $r\in\mathbb{Q}$. If $r=0$, write $r=-1-(-1)$. If $r<0$, write $r=2r-r$. If $r>0$, write $r=(-r)-(-2r)$. We see that $A-B$, in this case, contains $\mathbb{Q}$.

3. We show that $\sqrt{6}$ and $\sqrt{2}+\sqrt{3}$ are both irrational.

To show that $\sqrt{6} \notin \mathbb{R}$, prove by contradiction. If $\sqrt{6}$ were rational, we could write $\sqrt{6} = \frac{m}{n}$ where $m \in \mathbb{Z}, n \in \mathbb{N}$. By squaring both sides of this equation and multiplication by n^2, we see that $6n^2 = m^2$. Now consider prime factorization of $6n^2$ and m^2. We consider zero as an even number here. So $6n^2$ has an odd number of factors of 2, while m^2 has an even number of factors of 2. Thus $6n^2 \neq m^2$, a contradition.

To show that $(\sqrt{2}+\sqrt{3}) \notin \mathbb{Q}$, we use the previous result. Set $x = \sqrt{2}+\sqrt{3}$. Then $x^2 = 2 + 2\sqrt{6} + 3 \Rightarrow \frac{x^2-5}{2} = \sqrt{6}$.

We note that $\mathbb{Q}$ is field, so arithmetic is closed in $\mathbb{Q}$. So, $x \in \mathbb{Q} \Rightarrow \frac{x^2-5}{2} \in \mathbb{Q}$. Hence $\sqrt{6} \in \mathbb{Q}$, a contradiction.

4. We consider sequences of the form $\{s_1, s_2, \dots\}$ and determine which ones are Cauchy.

We know that the Cauchy Criterion for $\mathbb{R}$ holds (ie. $\mathbb{R}$ is a complete metric space): (s_n) is Cauchy $\Leftrightarrow$ (s_n) converges in $\mathbb{R}$

- $\{1, 1\frac{1}{2}, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}, \dots\}$ is <u>not</u> Cauchy:
  $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(\frac{n+1}{2}\right) = \infty$, so (s_n) diverges $\Rightarrow (s_n)$ is not Cauchy.

  Directly, set $\varepsilon = 1/3$. Given $N \in \mathbb{N}$, set $n = N+1$ and $m = N+2$. Then $n, m > N$, yet $|s_n - s_m| = \left|\frac{N+1}{2} - \frac{N+2}{2}\right| = \frac{1}{2} > \varepsilon$.

- $\{2, 2\frac{1}{4}, 2\frac{1}{9}, 2\frac{1}{16}, \dots\}$ is Cauchy:
  $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\left(2 + \frac{1}{n^2}\right) = 2$, so (s_n) is Cauchy.

  Directly, let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $1/N < \varepsilon/2 \Rightarrow 2/N < \varepsilon$
  For any $n, m > N$, we have $|s_n - s_m| = \left|\frac{1}{n^2} - \frac{1}{m^2}\right| \leq \frac{1}{n^2} + \frac{1}{m^2} < 2/N < \varepsilon$.

- $\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots\}$ is Cauchy:
  $\lim_{n\to\infty} (-1)^{n+1}/n = 0$, so (s_n) is Cauchy

  Directly, let $\varepsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $1/N < \varepsilon/2 \Rightarrow 2/N < \varepsilon$.
  For any $n, m > N$, $|s_n - s_m| = \left|\frac{1}{n} \pm \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < 2/N < \varepsilon$.

- $\{1, -1\frac{1}{2}, 1\frac{1}{3}, -1\frac{1}{4}, \dots\}$ is <u>not</u> Cauchy:
  Set $m_k = 2k$ and $n_k = 2k+1$ for $k = 1, 2, 3, \dots$
  Then $\lim_{k\to\infty} s_{m_k} = -1$ while $\lim_{k\to\infty} s_{n_k} = 1$. So (s_n) has two subsequential limits. Thus (s_n) does not converge $\Rightarrow (s_n)$ is not Cauchy

  Directly, Set $\varepsilon = 1$. Given $N \in \mathbb{N}$, set $n = N+1$ and $m = N+2$. Then n is odd $\Leftrightarrow$ m is even. Whence
  $$|s_n - s_m| = \left|(-1)^{N+2}\left(1 + \frac{1}{N+1}\right) - (-1)^{N+3}\left(1 + \frac{1}{N+2}\right)\right| = \left|\left(1 + \frac{1}{N+1}\right) + \left(1 + \frac{1}{N+2}\right)\right| > 1 = \varepsilon.$$

5. We consider two functions $\mathbb{R}^2 \to \mathbb{R}$, and determine which ones are inner products.

- $\langle (a,b),(x,y)\rangle = ax - by$ is <u>not</u> an inner product.

In particular, $\langle (0,1),(0,1)\rangle = -1$ which violates positivity.

In general, $\langle (a,b),(a,b)\rangle = a^2 - b^2 < 0$ whenever $|a| < |b|$.

- $\langle (a,b),(x,y)\rangle = 2ax + 3by$ is an inner product.
We check the conditions:

(Symmetry) $\langle (a,b),(x,y)\rangle = \langle (x,y),(a,b)\rangle$ is clear since $\mathbb{R}$ is commutative.

(Additivity)
$$\begin{aligned}\langle (a,b),(x,y)+(c,d)\rangle &= 2a(x+c) + 3b(y+d)\\ &= (2ax + 2ac) + (3by + 3bd)\\ &= (2ax+3by) + (2ac+3bd)\\ &= \langle (a,b),(x,y)\rangle + \langle (a,b),(c,d)\rangle\end{aligned}$$

(Scalers)
$$\begin{aligned}\lambda\langle (a,b),(x,y)\rangle &= \lambda(2ax + 3by)\\ &= 2a\lambda x + 3b\lambda y\\ &= 2(a\lambda)x + 3(b\lambda)y\\ &= \langle (a\lambda, b\lambda),(x,y)\rangle\\ &= \langle \lambda(a,b),(x,y)\rangle\end{aligned}$$

(Positivity) $\langle (a,b),(a,b)\rangle = 2a^2 + 3b^2 \geq 0$ is clear.
It is also clear that $2a^2 + 3b^2 = 0$ if and only if $a = b = 0$.

6. We use question 5 to prove the inequality

$$(2ax+3by)^2 \leq (2a^2+3b^2)(2x^2+3y^2).$$

Since $\langle (a,b),(x,y) \rangle = 2ax+3by$ defines an inner product on $\mathbb{R}^2$ (shown in previous problem), we may the norm $\|(a,b)\| = \sqrt{2a^2+3b^2}$.

The Cauchy-Schwartz inequality yields

$$(2ax+3by)^2 = |\langle (a,b),(x,y) \rangle|^2 \leq \|(a,b)\|^2 \cdot \|(x,y)\|^2$$

$$= (2a^2+3b^2)(2x^2+3y^2).$$