

Math 127 A 2

Winter quarter, 2005

Solutions to homework 2

Using cuts is a bit tricky, since it is often the case that we are trying to establish well known properties of the real numbers and so *cannot* assume these. So arguments are not easy to set out.

Problem 10,

We want to find the cut C corresponding to a real number y with the property that $xy = 1$, where x is a non-zero real number corresponding to some given cut A .

(a) In this case, $x > 0$ and so $y > 0$. We think of A as a set of rational numbers with the three cut properties and construct the cut C . In fact, by definition of multiplication of cuts,

$$AC = \{r \times s; r \in A, s \in C, r > 0, s > 0\} \cup \{r \in \mathbb{Q}, r \leq 0\}.$$

Note here we have to define the product in this slightly awkward way, to get the right set of rationals. Multiplying two large negative rationals gives a large positive rational, which we do not want in our product.

To complete the discussion, we now want to use the definition of product to describe the cut C . So since the product cut $AC = 1^*$, all the elements in $\{r \times s; r \in A, s \in C, r > 0, s > 0\}$ must be precisely the rationals in the interval $(0, 1)$. Hence we see that if the positive rationals in A lie in the interval $(0, x)$, then the positive rationals in the cut C must lie in the interval $(0, \frac{1}{x})$. We deduce that this describes C , or better, we can just say that the positive rationals in C are the reciprocals of the positive rationals outside of A , leaving out the smallest element of the complement. Hence $C = \{\frac{1}{r}; r > 0, r \in \mathbb{Q} \setminus A, \text{ but not the smallest element}\} \cup \{r \leq 0; r \in \mathbb{Q}\}$.

(b) This is very similar to (a) so not difficult. A nice solution is to describe the complements of the cuts, rather than the cuts themselves. So if $B = \mathbb{Q} \setminus A$ and $D = \mathbb{Q} \setminus C$, then the complement of the cut $\mathbb{Q} \setminus AB$ is $\{r \times s; r < 0, s < 0, r \in B, s \in D\} \cup \{r > 0; r \in \mathbb{Q}\}$. Therefore we see that $D = \{\frac{1}{r}; r < 0, r \in A\} \cup \{r > 0; r \in \mathbb{Q}\}$, with a largest element included, in the case of rational cuts. So C is the complement of this set, i.e. $C = \{s < 0, s \neq \frac{1}{r}; r < 0, r \in A\}$, leaving out the largest element.

Note that one can also work with the real ‘symbols’ as a guide. So here we again expect $y = \frac{1}{x}$. Hence the cut corresponding to $\frac{1}{x}$, where $x < 0$ would have rationals $r < \frac{1}{x}$. We can then see these are precisely the rationals $\{r < \frac{1}{s}; \text{ for every } s \in A\}$.

(c) Uniqueness. The issue here is again what do we really need to do. We have really done this in parts (a) and (b) already, by explicitly describing the cut corresponding to y . Another way of doing this would be to say, suppose there were two real numbers y, z with the properties that $x \times y = x \times z = 1$. We can then deduce that $x \times (y - z) = 0$. It is easy to deduce from the properties of ordering, that if $x > 0, y - z > 0$, then their product is also > 0 with the other three cases of positive and negative choices similar. So we deduce that $y - z = 0$, i.e $y = z$. Note that here we are using simple properties of addition, subtraction and order, but not multiplication or division.

Problem 12.

Here $b = \text{l.u.b}S$ for some non empty set S of real numbers.

(a) We need to show that given $\epsilon > 0$ we can find an element $s \in S$ with $b - \epsilon \leq s \leq b$. The usual way of tackling such questions is to try to prove this by contradiction. So suppose no such element s exists, for some particular choice of ϵ . Now for this choice, we get that for any element $s \in S$, that the inequality $b - \epsilon \leq s$ must be false. Hence this means that $s > b - \epsilon$. But if this were true, then clearly $b - \epsilon$ would be an upper bound for S smaller than b . This contradicts our assumption that b is the l.u.b and the inequality is proved. (The second part, that $s \leq b$ follows trivially since this is true for any upper bound of S).

(b) A very trivial example is where S has only one element! So $S = \{s\}$ and clearly then $b = s$ is the l.u.b of S . In this case we cant satisfy a strict version of the inequality.

(c) If $x = A$ is a cut, we want to show that x is the l.u.b of A . So to do this, we need to show that x is an upper bound and then it is the smallest upper bound. Let $r \in A$. By definition, $r < x$ and so x is indeed an upper bound for A . On the other hand, suppose that b is some other upper bound for A . We need to prove that $b \geq x$. A good tactic would be again to try to do this by contradiction. So let us assume that $b < x$. Hence by a result in class, there is a rational number r with $b < r < x$. But then by definition, $r \in A$ and this contradicts our assumption that b is an upper bound for A . So we conclude that x is indeed the l.u.b.

Problem 14.

Given $y \in \mathbb{R}$, a positive integer n and $\epsilon > 0$, find a $\delta > 0$ so that if $u \in \mathbb{R}$ satisfies $|u - y| < \delta$, then $|u^n - y^n| < \epsilon$.

The hint for this problem is to use the factorisation $u^n - y^n = (u - y)(u^{n-1} + u^{n-2}y + \dots + uy^{n-2} + y^{n-1})$. Note if you have not seen this factorisation before, it is a generalisation of the difference of squares formula, namely

$u^2 - y^2 = (u - y)(u + y)$. It is easy to verify the formula is true by just multiplying it out.

Now the question is how to use this factorisation. The idea is to write $|u^n - y^n| = |(u - y)|(u^{n-1} + u^{n-2}y + \dots + uy^{n-2} + y^{n-1})|$. By our assumption that we have some $\delta > 0$ to be determined, so that $|u - y| < \delta$. Hence we can plug this in and conclude that $|u^n - y^n| < \delta|(u^{n-1} + u^{n-2}y + \dots + uy^{n-2} + y^{n-1})|$. To get the right side $< \epsilon$, we need some control on the term $|(u^{n-1} + u^{n-2}y + \dots + uy^{n-2} + y^{n-1})|$. If we can bound this term, i.e prove that it is always $< C$ for some constant $C > 0$, then we can write $|u^n - y^n| < \delta C < \epsilon$, so long as $\delta < \frac{\epsilon}{C}$. So we will then have found the right choice for δ and concluded the argument.

There are two ways to bound the term - a direct method and an induction method, as suggested by the author. Let's do both.

For the direct method, use the triangle inequality to show that $|u^{n-1} + u^{n-2}y + \dots + uy^{n-2} + y^{n-1}| \leq (|u^{n-1}| + |u^{n-2}||y| + \dots + |u||y^{n-2}| + |y^{n-1}|)$. Note that $|y|$ is a constant, so we just need to worry about $|u|$. But we are also given that we are going to take u so that $|u - y| < \delta$ for some δ . Now we can always choose δ quite small, in particular at most 1. In this case, $|u - y| < 1$ and so writing $u = u - y + y$ gives $|u| \leq |u - y| + |y| < 1 + |y|$. We can then plug this in and estimate $|u^{n-1} + u^{n-2}y + \dots + uy^{n-2} + y^{n-1}| < (1 + |y|)^{n-1} + |1 + |y||^{n-2}|y| + \dots + (1 + |y|)|y^{n-2}| + |y^{n-1}|$. Now the right side is a mess but is certainly a constant so we can put it equal to C . This concludes the direct approach.

For the induction approach, start with $n = 2$. For this case, $u^2 - y^2 = (u - y)(u + y)$. Then the bound $|u^2 - y^2| = |u - y||u + y| < \delta(|u| + |v|) < \delta(2|y| + 1) = \delta C$ gives the result for this case. Exactly as in the previous argument, we need to use the estimate that $|u| < |y| + 1$. For the induction step, we write $u^{n-1} + u^{n-2}y + \dots + uy^{n-2} + y^{n-1} = u(u^{n-2} + u^{n-3}y + \dots + y^{n-2}) + y^{n-1}$. Hence $|u^{n-1} + u^{n-2}y + \dots + uy^{n-2} + y^{n-1}| \leq |u||u^{n-2} + u^{n-3}y + \dots + y^{n-2}| + |y^{n-1}|$. By the induction hypothesis, we can suppose that $|u^{n-2} + u^{n-3}y + \dots + y^{n-2}| < C'$. So again we get $|u^{n-1} + u^{n-2}y + \dots + uy^{n-2} + y^{n-1}| \leq |u|C' + |y^{n-1}| \leq (|y| + 1)C' + |y^{n-1}| = C$, where C is a new constant. So the result follows.

Problem 15.

Given a real number $x > 0$ and a positive integer n , we are required to show that there is a unique real number $y > 0$, so that $y^n = x$. The hint provided is to define $y = \text{l.u.b}\{s \in \mathbb{R}; s^n \leq x\}$. Then the idea is a proof by contradiction again. So let's assume that $y^n < x$. The idea is to show that since there is a gap of a certain size between y^n and x , we can add a small amount to y and still get a number whose n th power is at most x . But then this will contradict our assumption that $y = \text{l.u.b}\{s \in \mathbb{R}; s^n \leq x\}$, since y will not even be an upper bound for this set.

So let's try to add δ to y , where δ is going to be chosen in a minute. Then $s = y + \delta$ satisfies $s^n - y^n = (s - y)(y^{n-1} + sy^{n-2} + \dots + s^{n-2}y + s^{n-1})$, using the

factorisation trick of Q.14. Putting in $s = y + \delta$ and using the same idea as in Q.14, we find that $s^n - y^n < \delta C$ for some constant C independent of δ small. Hence by choosing $\delta C < x - y^n$, the conclusion is that $s^n - y^n < x - y^n$ and so s^n is in the interval (y^n, x) . This gives the contradiction we want, namely that y is not an upper bound for the set $\{s \in \mathbb{R}; s^n \leq x\}$. So this case is complete.

Next, let's examine the case where $y^n > x$. We want to do a similar argument to the previous one, but this time must use the fact that y is defined as an l.u.b, not just an upper bound. (A good thing to remember is that if some key information is given, then it should be used somewhere!).

So we need to somehow show that there is a smaller upper bound for $\{s \in \mathbb{R}; s^n \leq x\}$ than y . So the obvious thing to try is to show that $y - \delta$ is such an upper bound, for δ sufficiently small. We need to prove that given any s with $s^n \leq x$, then $s \leq y - \delta$. Again we will do this by contradiction. Suppose that for no δ is it true that for every $s, s \leq y - \delta$. So we can find a sequence say $\frac{1}{m}$ of δ values getting smaller and smaller and for each of these, a real number s_m so that $s_m + \frac{1}{m} > y$ for every m . We also will need to use the key assumptions that $y^n > x$ and $s_m^n \leq x$. But then, it follows that $(s_m + \frac{1}{m})^n > y^n$ for every m . Combining this with $s_m^n \leq x$ gives $(s_m + \frac{1}{m})^n - s_m^n > y^n - x$ for every m . But it is easy to show this is impossible, by a very similar argument to our previous ones. The left side can be estimated as usual by being at most $\frac{1}{m}C$ where C is a constant, and so gets arbitrarily small. Hence it cannot always be larger than the fixed right side. This completes the proof for this case.

The last part is to prove that y is unique. Note we have constructed a y but possibly there could be a z satisfying $z^n = x$ with $z \neq y$. But it is easy to use the same method yet again to rule this out. For if we assume say $y > z > 0$ and look at $y^n - z^n$ and factorise it, we see it is a product of positive numbers and hence is positive. The case where $y < z < 0$ is very similar. Finally the case where $y < 0 < z$ and n is odd is trivial, since then one power is positive and one negative, so they cannot be equal. On the other hand, if n is even, we can replace the problem by studying y^2, z^2 and a power k where $n = 2k$ to reduce to the first case.