Math 127 A 2

Winter quarter, 2005

Solutions to homework 6

Problem 10 (a) To do this, we want a 'gap' between the sphere of radius r and the open ball of radius r, so that they are not 'adjacent'. But note we can remove points and still have a metric space. So for example, we can take the closed disk of radius r centred at the origin in the plane R^2 and remove all the points of distance strictly between r and $\frac{r}{2}$ from the origin. So we have $M = \{(x, y) : x^2 + y^2 \leq \frac{r^2}{4}\} \cup \{(x, y) : x^2 + y^2 = r^2\}$. Then the open ball of radius r is $M_r = \{(x, y) : x^2 + y^2 \leq \frac{r^2}{4}\}$ and this is a clopen set. The sphere of radius r is not in the closure of M_r.

(b) The boundary must be contained in the sphere. For the open ball of radius r has boundary outside the ball (which is its own interior - being open). Also a limit of a sequence of points in the open ball is easily seen to be in the closed ball, hence the boundary $\partial M_r = \bar{M_r} \setminus int M_r$ is contained in the sphere of radius r.

Problem 14 (a) p is a limit of S if and only if $d(p, S) = 0$. For suppose x_n in S converges to p. Then $d(x_n, p)$ converges to zero and hence $d(p, S) = inf\{d(p, x) : x \in S\} = 0$.

Conversely if $d(p, S) = inf\{d(p, x) : x \in S\} = 0$, then we can find a sequence of points x_n so that $d(x_n, p)$ converges to zero. So p is a limit point of S.

(b) To show that $p \rightarrow d(p, S)$ is a uniformly continuous function, we need to show that for any $\epsilon > 0$ there is a $\delta > 0$ so that if $d(p, q) < \delta$, then $|d(p, S) - d(q, S)| < \epsilon$. In fact, it suffices to choose $\delta = \epsilon$. For if $d(p, q) < \epsilon$ and $d(p, S) = d$, then there is a sequence of points x_n in S with $d(p, x_n)$ converging to d. But then by the triangle inequality, $d(q, x_n) \leq d(p, q) + d(p, x_n)$. The right side converges to $d + d(p, S)$ and so we conclude that $d(q, S) = inf\{d(q, x) : x \in S\} < \epsilon + d(p, S)$. The roles of p, q can be interchanged to prove $d(p, S) \leq \epsilon + d(q, S)$. Putting these together gives that $|d(p, S) - d(q, S)| < \epsilon$.

Problem 27 (a) We want to show that $\cap A_n$ consists of precisely one point. So there are two issues - to show there is at least one point in the intersection and that there cannot be more than one point. For the first, we proceed as in the proof of the same result for compact sets. We choose a point x_n in each closed set A_n. Since the diameters of the sets A_n converge to zero, it is easy to check that x_n is a Cauchy sequence. (All the points in the sequence for $n > N$ lie in the closed set A_N which can be assumed to have diameter $< \epsilon$ and so are distance within ϵ of each other.) But then by completeness, we get that x_n converges to x. But then since the sets A_n are closed, the point x is in the closure of A_n and so is in A_n. But then we have found a point in the intersection of all the sets. Showing there cannot be more than one point is easy; if there are points x, y in all A_n then there is a lower bound for the diameters of the A_n, namely $d(x, y)$. Hence the diameters cannot converge to zero.

(b) The example where $A_n = [n, \infty)$ give closed sets which are nested in a complete space, but their diameter does not converge to zero. So the intersection is empty in this case.

Problem 29 (a) A convergent sequence is easily seen to be bounded. For if x_n converges to x, then we know that given $\epsilon > 0$ there is N so that if $n > N$ then $d(x_n, x) < \epsilon$. Hence all the points x_n for $n > N$ are in the ball $M_\epsilon(x)$. There are only finitely many other points in the sequence so we can find a ball of radius max $\{d(x_1, x), \ldots d(x_N, x), \epsilon\}$ about x containing the whole sequence.

(b) The same will be true for a Cauchy sequence, since $d(x_n, x_m) < \epsilon$ for all $n, m > N$ implies that all the points x_m for $m > N$ are within ϵ of distance to x_{N+1}. So the same argument as in (a) applies.

Problem 38 We want to prove that the closed unit ball B for any norm on R^n is compact, i.e closed and bounded, by Heine Borel. This is quite tricky - I should have given a hint as to how to tackle it. There are various slightly different approaches. One thing to note which we did show in class is that B is convex (this is just the triangle inequality for norms). So this should help.

A key idea is to try to prove that the map $x \to ||x||$ is continuous. For once this is established, the norm $||||$ on the unit sphere S^{n-1} in R^n must have a bounded image, since the continuous image of a compact set is compact, hence closed and bounded. But then the unit ball B is bounded, since any vector x has $||x|| = ||\frac{x}{|x|}|||x|$ and so if the minimum value of $||||$ on the unit sphere is b, then the maximum distance of a point of B from the origin is $\frac{1}{b}$.

Similarly to show that B is closed, we need to show that if a sequence of points x_i in B is convergent to a limit x, then $x \in B$. But this follows immediately from continuity of $||||$.

On the other hand, it is easy to see that $||||$ is continuous, if B is bounded! For suppose we have a sequence of points x_i converging to x. Then the Euclidean distance $|x_i - x|$ converges to zero. But then the norms $||x_i - x||$ converge to zero also, since it is easy to see that B bounded is the same as saying that the ratio $\frac{||x||}{|x|}$ is bounded by a constant for all points $x \in R^n$. So we somehow have to show boundedness of B *before* we can show that $\|\|$ is continuous.

Since B is convex, if it is unbounded, then we will show there is a 'ray' i.e a set of vectors of the form sx for all $s \geq 0, s \in R$. It is then easy to get a contradiction, since most of the ray will be outside of B. To make life easier, let's picture the case where $n = 2$. We start with the observation that there must be points $(a, 0), (0, b)$ with $a, b > 0$ which have norm 1. Hence the convex combinations of the points $(a, 0), (0, b), (-a, 0), (0, -b)$ will lie in B by convexity. These form a 'diamond' shape. If B was unbounded then there would be a sequence of points x_i in B with $|x_i|$ converging to ∞. The points $\frac{x_i}{|x_i|}$ are in the unit sphere S^{n-1} which is compact, so there is a convergent subsequence x_{i_k} to a point x in S^{n-1}. We claim that the ray $\{sx\}$ is completely inside B. Thinking about the case $n = 2$ helps a lot. Notice that all convex combinations of points in the diamond shape together with each point x_i is inside B by convexity. But as the points converge to the direction of the ray, we see that there will be points in these convex combinations in the ray and further and further out. (Elementary Euclidean geometry!). So we have a sequence of points in B in the ray converging to ∞. This shows that the whole ray is indeed inside B.

But if the ray is inside B, then no positive multiple of x can have norm 1, which is clearly a contradiction. The same argument works in all dimensions by replacing the diamond by the appropriate n-dimensional version of it.

This is a very tough problem, but it has lots of very important consequences. One can also interpret this result as saying that any two norms on R^n give distances which are comparable. So all norms induce the same topology on R^n etc.

Problem 39 If $A \times B$ is compact in $M \times N$ we can either use the sequential definition of compactness to check that A and B are compact, or we can use an easier argument, that the projections $p_1 : M \times N \to M$ and $p_2 : M \times N \to N$ are continuous. But then the continuous image of a compact set is compact, so $p_1(A \times B) = A$ and $p_2(A \times B) = B$ are both compact.