Math 127 A 2

Winter quarter, 2005

Solutions to homework 7

Problem 43

If K_i are compact with diameter $\geq \mu$ then we want to prove that $\cap K_i$ has diameter $\geq \mu$. The argument is as follows - choose points $x_i, y_i \in K_i$ so that $d(x_i, y_i) \geq \mu$. Such a pair must exist by definition of diameter as the supremum of the distance between pairs of points in a set. But now by compactness, since the K_i are nested, the sequences $\{x_i\}$ and $\{y_i\}$ are both in K_1 and so have convergent subsequences $\{x_{n_i}\}$ and $\{y_{m_i}\}$ with limits x and y in K_1. To make life easier later on, we can improve the situation by choosing first the subsequence $\{x_{n_i}\}$ and then choosing $\{y_{m_i}\}$ as a subsequence of $\{y_{n_i}\}$. Then we can go back and replace $\{x_{n_i}\}$ by the subsubsequence $\{x_{m_i}\}$, so that both subsequences have exactly the same indices. This will prove very useful at the end.

But it is easy to see that x and y are actually in all the nested sets K_i, since the subsequences are in K_i after a finite number of terms, so by closure of each K_i, the limits x and y must be in K_i. Finally we observe that $d(x_i, y_i) \geq \mu$ for each i implies that $d(x, y) \geq \mu$, since the subsequences $\{x_{m_i}\}$ and $\{y_{m_i}\}$ converge to x, y. (Note we are using here that the distance function is actually continuous from $M \times M \to \mathbb{R}$ which is proved in the book and is simple to verify directly.

Problem 45

An example of a non-compact space with positive Lebesgue number for every open covering is N with the discrete metric. For $d(x, y) = 1$ if $x \neq y$. So if we take the balls of radius 1 about every point, we get precisely that point. Hence these balls will lie inside the open sets of any open cover and so we see that the Lebesgue number must be at least 1 in this case.

Problem 55

(a) Is the closure of a disconnected space disconnected? One should expect the answer to be no, since two pieces can 'join up' when the closure is taken. A simple example is $S = (0, 1) \cup (1, 2)$ inside $\mathbb{R}$. Then $\bar{S} = [0, 2]$ which is connected.

(b) What about the interior of a disconnected set? Well again the answer should be no, since sets can have empty interior. A very simple example $S = \{0, 1\}$ in $\mathbb{R}$ is disconnected and has empty interior.

Problem 59

To prove that the annulus $A = \{z : r \leq |z| \leq R\}$ is connected, it is easiest to prove it is path connected. Given two points z_1, z_2 we can join them by a path by going from z_1 to cz_1 where $c = \frac{|z_2|}{|z_1|}$ and then going around the circle of radius $|z_2|$ from cz_1 to z_2.

Problem 69

(a) The best to show that $M \times N$ is connected, if both M and N are connected, is to observe that if $M \times N = A \cup B$ where A, B are disjoint open non-empty sets, then every subset $\{x\} \times N$ and $M \times \{y\}$ must be either in A or in B, for $x \in M$ and $y \in N$. The reason is that $A \cap (\{x\} \times N)$ and $B \cap (\{x\} \times N)$ are open and disjoint and since $\{x\} \times N$ is connected (it is clearly homeomorphic to N, we see that one or other of these sets must be empty. But then it is easy to see that either A or B must be empty since if some $\{x\} \times N$ is in A and some $\{x'\} \times N$ is in B it would follow that $M \times \{y\}$ is in neither A nor B as $x \times y$ would be in A and $x' \times y$ would be in B.

(b) If $M \times N$ is connected, it is easy to see that M, N must both be connected since they are continuous images of $M \times N$ under the projection maps π_1, π_2 from $M \times N$ onto M or N.

(c) Path connectedness is pretty easy - if there are paths f, g in M, N from x to x' and from y to y', then $f \times g$ is a path from $x \times y$ to $x' \times y'$. Conversely again the projections π_1, π_2 show that if $M \times N$ is path connected, so are M, N. (A continuous image of a path connected space is path connected).

Problem 71

If A, B are non-empty disjoint compact spaces, then we can find points $a_0 \in A, b_0 \in B$ so that $d(a_0, b_0) \leq d(a, b)$ for all $a \in A, b \in B$. The reason is that clearly we can find sequences $a_n \in A, b_n \in B$ so that $d(a_n, b_n)$ converges to the infimum of all distances between pairs of points in A, B. Now by compactness, we can choose a subsequence of the a_n converging to a_0 in A and a further subsequence of the corresponding subsequence of the b_n converging to a point b_0 in B. But now it is easy to see by continuity of d that $d(a_0, b_0)$ is equal to this infimum.

Notice this shows the interesting fact that if the infimum is zero, then A, B would have to intersect. Hence the infimum must be > 0 is A, B are disjoint.

Remark

Continuity of d gets used a lot. This says that if $x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x, y)$. To check this, note that $|d(x, y) - d(x_n, y_n)| = |d(x, y) - d(x, y_n) + d(x, y_n) - d(x_n, y_n)| \leq |d(x, y) - d(x, y_n)| + |d(x, y_n) - d(x_n, y_n)|$. Now by the triangle inequality, $|d(x, y) - d(x, y_n)| \leq d(y, y_n) \leq \epsilon$ for n large and similarly $|d(x, y_n) - d(x_n, y_n)| \leq d(x, x_n) \leq \epsilon$ for n large. We conclude that $d(x_n, y_n) \to d(x, y)$.