Math 127 A 2

Winter quarter, 2005

Mid term exam 2

Throughout this exam, you may assume the fact that if $f : (a, b) \to \mathbb{R}$ is a differentiable function, then f is continuous.

Question 1

Suppose that $\{x_n\}$ is a sequence of real numbers which converges to the real number x. Prove that $\{{x_n}^2\}$ is also a convergent sequence with limit x^2. Is it true or not that the sequence $\{\frac{1}{x_n}\}$ converges to $\frac{1}{x}$? Briefly justify your answer.

Question 2

If $f : M \to \mathbb{R}$ and $g : M \to \mathbb{R}$ are continuous functions, prove that $\max\{f, g\}$ is a continuous function from M to $\mathbb{R}$. Deduce that $|x| : \mathbb{R} \to \mathbb{R}$ is continuous.

Question 3

If U is an open subset of a metric space M, prove that with the induced metric from M, a subset V is open in U if and only if V is open in M. Give an example to show that this fails if U is not open in M, i.e find an example where U is not open in M, and there is a subset V of U which is open in U but not in M.

Question 4

Which of the following pairs of metric spaces are homeomorphic? Give a brief justification for your answers.

- $\mathbb{R}^2$ and $\mathbb{R}$, with the standard Euclidean metrics.
- $B = \{(x, y) : x^2 + y^2 \leq 1\}$ and $D = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ as subspaces of $\mathbb{R}^2$ with the standard Euclidean metric.
- $\mathbb{Q}$ and $\mathbb{R}$ with the standard Euclidean metrics.
- $(0, 1)$ and $\mathbb{R}$ with the standard Euclidean metrics.

Question 5

Give an example of a countably infinite subset S of $\mathbb{R}$ which has the discrete topology as a subspace. Give a second example of a countably infinite subset S' of $\mathbb{R}$ which has a subspace topology for which every point $x \in S'$ is a cluster point of S'. Give a brief justification of your answers.

Question 6

Let $S = [\frac{1}{2}, 1] \cup [\frac{1}{8}, \frac{1}{4}] \cup \cdots \cup [\frac{1}{2^{2k+1}}, \frac{1}{2^{2k+2}}] \cup \ldots$. Find the closure, the interior and the boundary of S as a subset of $\mathbb{R}$. Give a brief justification of your answer.