MAT 127A-2 Midterm 2 Winter 2005

Problem 1

Suppose x_n converges to x. Consider $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=x^2$. This is just the squaring function which has derivative $f'(x)=2x$. So $f(x)$ is continuous.

Thus $\lim_{n\to\infty} x_n^2 = \lim_{n\to\infty} f(x_n) = f\left(\lim_{x\to\infty} x_n\right) = f(x) = x^2$.

Now consider the sequence $\frac{1}{x_n}$, where each x_n is nonzero.

If $\lim_{n\to\infty} x_n = x \neq 0$, then $g(x) = 1/x$ is differentiable, thus continuous (on the domain $(x-\varepsilon, x+\varepsilon)$ for ε small enough) with derivative $g'(x) = -1/x^2$. Thus $\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{x}$.

Problem 2

Suppose $f: M\to\mathbb{R}$ and $g: M\to\mathbb{R}$ are continuous. Define a function $h: M\to\mathbb{R}$ by $h(x)=\max\{f(x),g(x)\}$.

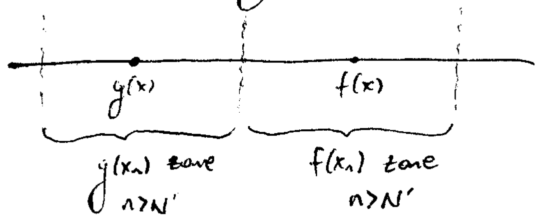
We will show that h is continuous. Let $x\in M$. To show that h is continuous at x, we use sequences. Suppose that $\{x_n\}\subseteq M$ is a sequence with $\lim_{n\to\infty} x_n = x$.

Let $\varepsilon>0$. Choose $N\in\mathbb{N}$ large enough so that $n>N \Longrightarrow |f(x_n)-f(x)|<\varepsilon$ and $|g(x_n)-g(x)|<\varepsilon$ (using continuity of f and g at x).

If $f(x)=g(x)$, then $|h(x_n)-h(x)|$ is $|f(x_n)-f(x)|$ or $|g(x_n)-g(x)|$, depending on n. In either case, $n>N \Rightarrow |h(x_n)-h(x)|<\varepsilon$.

Now suppose $f(x)\neq g(x)$. Let's say $f(x)>g(x)$. Set $\Delta = f(x)-g(x)$. Then choose $N'\geq N$ so that $N'\in\mathbb{N}$ and for all $n>N'$, we have $|f(x_n)-f(x)|<\Delta/2$ and $|g(x_n)-g(x)|<\Delta/2$.

Consequently $f(x_n) > g(x_n)$ for all $n > N'$:



Therefore $h(x_n) = f(x_n)$ $\forall n > N'$ $\Rightarrow |h(x_n) - h(x)|$

$= |f(x_n) - f(x)| < \varepsilon$ for $n > N'$.

So $\lim_{n \to \infty} h(x_n) = h(x)$. Thus h is continuous at x.

To show that $|x| : \mathbb{R} \to \mathbb{R}$ is continuous, just realize this function as $\max\{-x, x\}$, and apply the previous result.

<u>Problem 3</u> (<u>Notation</u>: if $x_0 \in M$ & $\varepsilon > 0$, $M_\varepsilon(x_0) = \{x \in M : d(x, x_0) < \varepsilon\}$)

We show that $V \underset{\text{open}}{\subseteq} U \iff V \underset{\text{open}}{\subseteq} M$.

Assume $V \underset{\text{open}}{\subseteq} U$. Let $x_0 \in V$. We show that $\exists\, \varepsilon > 0$ such that $M_\varepsilon(x_0) \subseteq V$. Since $V \underset{\text{open}}{\subseteq} U$, $\exists\, \delta > 0$ such that $M_\delta(x_0) \cap U \subseteq V$.

Since $U \underset{\text{open}}{\subseteq} M$, $\exists\, \varepsilon > 0$ such that $M_\varepsilon(x_0) \subseteq U$. Now set $\varepsilon' = \min\{\delta, \varepsilon\}$. Then $M_{\varepsilon'}(x_0) \subseteq V$. Thus $V \underset{\text{open}}{\subseteq} M$.

Conversely, suppose $V \underset{\text{open}}{\subseteq} M$. Let $x_0 \in V$. Then $\exists\, \delta > 0$ such that $M_\delta(x_0) \subseteq V$. Thus $M_\delta(x_0) \cap U \subseteq V \cap U = V$. Thus $V \underset{\text{open}}{\subseteq} U$. This part of the proof can actually be done purely topologically.

For the example, set $M = \mathbb{R}$ with usual metric, $U = [0,1]$, and $V = [0,1)$.

Problem 4

$\boxed{\mathbb{R} \text{ and } \mathbb{R}^2 \text{ are not homeomorphic}}$: $\mathbb{R}$ - origin is torn into two pieces while $\mathbb{R}^2$ - (a point) is still "connected".

$\boxed{B \text{ and } D \text{ are homeomorphic}}$ In terms of formulas, consider radial projection $f: \partial D \to \partial B$ with formula $f(x) = \frac{x}{\|x\|}$. This is clearly a homeomorphism. Let $g = f^{-1}: \partial B \to \partial D$. To extend g to all of B, define $g(0) = 0$ and map segments accordingly: $\forall r \in [0,1]$ and $v \in \partial B$, set $g(rv) = r \cdot g(v)$.

$\boxed{\mathbb{Q} \text{ and } \mathbb{R} \text{ at not homeomorphic}}$, since they are not even of the same cardinality.

$\boxed{(0,1) \text{ and } \mathbb{R} \text{ are homeomorphic}}$ First note that $f: (0,1) \to (-\pi/2, \pi/2)$ defined by $f(x) = \pi x - \pi/2$ is a homeomorphism. Composing this with $\tan: (-\pi/2, \pi/2) \to \mathbb{R}$ gives the desired homeomorphism from $(0,1) \to \mathbb{R}$, namely $\tan(\pi x - \pi/2)$.

Problem 5

Any infinite subset of $\mathbb{Z}$ can be used for S. Note that $\forall n \in S$, $(n - 1/2, n + 1/2) \cap S = \{n\}$.

We should use $\mathbb{Q}$ for S'. Note that for any $\varepsilon > 0$ and $s \in \mathbb{Q}$, $(s - \varepsilon, s + \varepsilon) \cap \mathbb{Q}$ is infinite.

Problem 6

$\boxed{\bar{S} = \{0\} \cup S}$

To see that $0 \in \bar{S}$, let $\varepsilon > 0$ be given. Since $\lim_{k \to \infty} \frac{1}{4^k} = 0$, $\exists\, k \in \mathbb{N}$ s.t. $1/2^{2k} < \varepsilon$.
Thus $[0, \varepsilon) \cap S \neq \emptyset$. So $0 \in \bar{S}$.

To see that 0 is the only point in $\bar{S} - S$, let $x \in (0,1)$. If $x \notin S$, then $\exists\, k \in \mathbb{N} \cup \{0\}$ such that $x \in \left(\frac{1}{2^{2k}}, \frac{1}{2^{2k+1}}\right)$, since $\lim_{k \to \infty} \frac{1}{2^k} = 0$. Set $U = \left(\frac{1}{2^{2k}}, \frac{1}{2^{2k+1}}\right)$. Then U is a neighborhood of x that does not intersect S. Thus $x \notin \bar{S}$.

$\boxed{\mathrm{int}(S) = \left\{x \in \left(\frac{1}{2^{2k+1}}, \frac{1}{2^{2k}}\right) \text{ for some } k \in \mathbb{N} \cup \{0\}\right\}}$

In other words, $\mathrm{int}(S) = \bar{S} - \{\text{endpoints of intervals } [\frac{1}{2^{2k+1}}, \frac{1}{2^{2k}}]\}$.
This is clear.

$\boxed{\partial S = \{0, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^k}, \ldots\}}$

We just calculate $\partial S = \bar{S} - \mathrm{int}(S)$ from our previous answers.