Math 127 A 2

Winter quarter, 2005

Solutions to practice questions

1 The set C is not a cut since it does not contain negative rational numbers which are arbitrarily big. So it does not contain all the rationals less than any given one in the set. Another problem is that we could take a negative q' in B away from a positive q in A and get a number which is too large and positive.

To fix this up, note that we want C to ONLY have negative numbers, starting at $\sqrt{2} - sqrt3$. So we need to consider positive rationals q' with the property that $q'^2 > 3$ and take these away from positive rationals with the property that $q^2 < 2$ to define $\bar{C} = \{q - q'\}$.

2 The first is a norm, it has the standard properties
- $|\lambda v| = |\lambda||v|$ where v is a vector and λ a real number or scalar.
- $|v| \geq 0$ and $|v| = 0$ if and only if $v = 0$.
- $|v + w| \leq |v| + |w|$.

The second is not a norm since the first property is not satisfied.

3 Finite sequences of integers is a countable set. For if we take a fixed n, a sequence of n integers is the same as taking $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \cdots \times \mathbb{Z}$, ie a Cartesian product of n copies of $\mathbb{Z}$. But we proved in lectures, or from the book that a product of two countable sets is countable, so we can then repeat this n times (or use induction) to see that a product of n countable sets is countable.

But we also know that a countable union of countable sets is countable, again from class or from the book. So we can take the union of the sequences of length 1, the sequences of length 2, etc and get a countable union of countable sets which is countable.

The numbers in $[0, 1]$ with decimal expansions only involving $1, 2$ is an uncountable set. The reason is that we can use Cantor's diagonal argument. If we could list this collection, then we can change the first term from a 1 to a 2 or vice versa and so on down the diagonal.

4 This is a compact subset of $\mathbb{R}^2$ for example, since by Heine Borel, if we can show it is closed and bounded. Now it is certainly bounded, since if

$x^2 + 3y^2 = 1$ then $x^2 + y^2 \leq 1$ and so the region is inside the ball of radius 1 centered at the origin.

To prove it is closed, the easiest method is to see that there is a continuous function $f : (x, y) \to x^2 + 3y^3$ and the set is the inverse image of 1. But for a continuous function, the inverse image of a closed set is closed and one point is a closed set in $\mathbb{R}$. Alternatively it is easy to see that if a sequence (x_n, y_n) converges to (x, y) and ${x_n}^2 + 3y_n^2 = 1$ then $x^2 + 3y^2 = 1$ (This is just continuity of the function again!).

This set is the boundary of the set $\{(x, y) : x^2 + 3y^2 \leq 1\}$ again by continuity of the function, or by drawing the picture!

A single point can be the boundary of a subset of $\mathbb{R}^2$. Note that if we remove a single point from $\mathbb{R}^2$, then we have an open set, since it is the complement of a closed set. Hence the interior of this set is itself. On the other hand, the boundary is the closure minus the interior and the closure is clearly $\mathbb{R}^2$. So this is indeed an example where the boundary is one point.

An easier example is a single point also works, since the interior is empty. Since a single point is closed, we see that the closure minus the interior is just the point.

5 The graph of a continuous function $\mathbb{R} \to \mathbb{R}$ is connected, since it is the continuous image of a connected set, using the fact that $\mathbb{R}$ is connected. So this part is easy.

But to show that if a graph is connected, the function is continuous is hard. In fact, if you try to do this, no method seems to work. So you might suspect something is wrong. If you think then of the example in class of the topologists sine curve, this is almost the graph of a function and we proved it is connected. So you want to construct an example like the topologists sine curve which is a graph. This is a bit weird - an example is to take two sine curves, $sin\frac{1}{|x|}$ for $x \neq 0$. This nearly works but it is not a graph of a function on $\mathbb{R}$ since it is not defined yet at $x = 0$. Now we can add in the extra point, e.g by taking $f(0) = 0$. With this definition, it is pretty easy to show by the argument that if S is connected, and $S \subset T \subset \bar{S}$ then T is connected that the graph is now connected. For the graph on the positive x axis plus the origin is connected by the theorem as is the graph on the negative x axis plus the origin. So we have two connected sets with a point in common, which is then connected. It is rather obvious that the function is not continuous! This is very tricky - just an idea of how to proceed is all that I would be looking for here.

6 Completeness is convergence of Cauchy sequences to limits in the space. So if we have any compact subset of a complete metric space, we know that Cauchy sequences have limits and since a compact subset is closed, this will also be complete. (Actually this shows it is enough to have a closed subset of a complete space to be complete, ie compactness is more than we need). So the first and third examples are complete. The second example is more tricky. Suppose that we have a compact subset of a metric space and a Cauchy sequence in it. By compactness, there is a convergent subsequence to a limit in the space. But if a Cauchy sequence has a convergent subsequence, it is easy to see the whole sequence converges. So we see that the second example is also complete. Finally the last example is easily seen to be incomplete as we can take a sequence of irrationals converging to a rational.

Remarks

Note that some of these are quite hard to get all the details down. But on the other hand, if you understand the questions often you can work out the correct answers with only partial justification. This is the major thing for the most difficult parts. This practice exam is a bit more difficult than the final.