

# Math127- Summary of metric spaces

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A metric space  $M$  is a set with a distance function  $d$  satisfying

- $d(x, y) = d(y, x)$
- $d(x, y) \geq 0$  and  $d(x, y) = 0$  if and only if  $x = y$
- $d(x, z) \leq d(x, y) + d(y, z)$

The basic example is a normed vector space, ie  $V, \|\cdot\|$  where  $V$  is a vector space and  $\|\cdot\|$  is a norm. Then the distance  $d$  is given by  $d(x, y) = \|x - y\|$ .

Another important way of constructing metric spaces is by taking subspaces of a given metric space. So if  $M$  is a metric space and  $S$  is any subset, then  $S$  is a new metric space. We just define  $d(x, y)$  for  $x, y \in S$  by the same distance value as in  $M$ .

In a metric space, we have convergent sequences, exactly as in  $\mathcal{R}$ . A sequence of points  $x_i$  in  $M$  is convergent with limit  $x$  if  $d(x_i, x) \rightarrow 0$ . Equivalently, for every  $\epsilon > 0$ , there is an  $N$  so that if  $i > N$ , then  $d(x_i, x) < \epsilon$ .

Convergent sequences have unique limits and any subsequences have the same limit. An important use of convergent sequences is a characterisation of continuous maps.

A map  $f : M \rightarrow P$  is continuous, between metric spaces  $M$  and  $P$  at a point  $x \in M$ , if for every  $\epsilon > 0$ , there is a  $\delta$  so that if  $d(x, y) < \delta$ , then  $d(f(x), f(y)) < \epsilon$ . We say then that  $f$  is continuous, if it is continuous at every point  $x \in M$ .

Then an equivalent way of testing if  $f$  is continuous at  $x$  is that for any sequence  $x_i \rightarrow x$  in  $M$ , then  $f(x_i) \rightarrow f(x)$  in  $P$ .

An application of the sequence approach to continuity is that the composition of continuous functions is continuous. So if  $f : M \rightarrow P$  and  $g : P \rightarrow W$  are continuous maps between metric spaces, then  $g \circ f : M \rightarrow W$  is continuous. (Actually we can also show that if  $f$  is continuous at  $x$  and  $g$  is continuous at  $f(x)$ , then  $g \circ f$  is continuous at  $x$ .)

Two metric spaces are homeomorphic if there is a map between them which is continuous, a bijection and the inverse is continuous as well. This is the primary idea of equivalence of metric spaces. A standard way of proving that two metric spaces are homeomorphic, where they are both subspaces of the same vector space, is by radial projection. So for example, again region strictly inside a convex polygon in the plane is homeomorphic to the open unit disk. For we can shrink the polygon down to lie within the unit disk. Radial projection from the origin then gives an invertible continuous map between the two spaces.

Note that homeomorphism is easily seen to be an equivalence relation, since the composition of homeomorphisms is a homeomorphism.

A set  $U$  in a metric space is called *open* if for every  $x \in U$ , there is a radius  $r$  so that  $M_r(x)$  is inside  $U$ . Note that  $M_r(x)$  is the ball of radius  $r$  centered at

$x$ , namely  $\{y : d(x, y) < r\}$ . Key properties of open sets are:

- $\emptyset, M$  are open.
- the arbitrary union of open sets is open.
- the finite intersection of open sets is open.

An important example of an open set is an open ball, ie  $M_r(x)$  for any radius and center.

Similarly a set  $F$  in a metric space  $M$  is closed if it contains all its limit points, i.e limits of every convergent sequence of points of  $F$ . Then closed sets satisfy;

- $\emptyset, M$  are closed.
- the finite union of open sets is closed.
- the arbitrary intersection of open sets is closed.

A key result about closed sets is a set is closed if and only if it is the complement of an open set. A typical example of closed sets is  $\text{lim } S$ , which is all limit points of any set  $S$  in a metric space. So  $F$  is closed exactly when  $F = \text{lim } F$ . Another very useful result about closed sets, is that given any subset  $S$  we can define its *closure*  $\bar{S}$  by taking the intersection of all closed sets containing  $S$ . So  $\bar{S}$  is the 'smallest' closed set containing  $S$ . It is true that  $\text{lim } S = \bar{S}$ , so we can also construct the closure by taking all limit points.

The *interior* of a set  $S$  is defined as the largest open set contained in  $S$ . Then  $\text{int } S$  can be constructed by taking the union of all open sets contained inside  $S$ . Finally the boundary  $\partial S$  of a set  $S$  is defined as  $\bar{S} \setminus \text{int } S$ . Then  $\partial S$  is closed, since it can also be written as  $\bar{S} \cap (M \setminus \text{int } S)$ , ie the intersection of closed sets.

An important use of open and closed sets is to give another characterisation of continuity. Namely  $f : M \rightarrow P$  is continuous if and only if  $f^{-1}(F)$  is closed in  $M$  for every closed set  $F$  in  $P$  if and only if  $f^{-1}(U)$  is open in  $M$  for every open set  $U$  in  $P$ . As a typical application, if  $f : M \rightarrow \mathcal{R}$  is continuous, then  $\{x : f(x) < a\}$  and  $\{x : f(x) > a\}$  are open in  $M$ . Similarly  $\{x : f(x) \leq a\}, \{x : f(x) = a\}$  and  $\{x : f(x) \geq a\}$  are closed in  $M$ .

Open sets in  $\mathcal{R}$  can be completely characterised as finite or countably infinite unions of disjoint open intervals. Closed sets in  $\mathcal{R}$  are not so simple - basic examples are closed intervals  $[a, b]$  and isolated points  $\{a\}$  or finite collections of points. However we will find later much more complicated closed sets called Cantor sets.

### Practice examples

1 Show that if  $(M, d)$  and  $(P, d')$  are metric spaces, then so is the Cartesian product  $M \times P$ . Note that the metric on  $M \times P$  can be defined in several different ways

- $D(x_1, y_1), (x_2, y_2) = d(x_1, x_2) + d'(y_1, y_2)$  or
- $D(x_1, y_1), (x_2, y_2) = \max \{d(x_1, x_2), d'(y_1, y_2)\}$
- $D(x_1, y_1), (x_2, y_2) = \sqrt{(d(x_1, x_2))^2 + (d'(y_1, y_2))^2}$

Each way works and we will examine the differences and similarities later on.

2 In  $\mathcal{R}^2$  show that the set  $\{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$  is closed. Show that the set  $\{(x, y) : -1 < x < 1, -1 < y < 1\}$  is open. Finally show that the boundary of either of these sets is the set  $\{(x, y) : x = 1, -1 \leq y \leq 1; x = -1, -1 \leq y \leq 1; y = 1, -1 \leq x \leq 1; y = -1, -1 \leq x \leq 1\}$ .

3 Show that if  $f : M \rightarrow \mathcal{R}$  and  $g : M \rightarrow \mathcal{R}$  are continuous functions then  $f + g : M \rightarrow \mathcal{R}$  is continuous.

4. Use a continuous function to show that  $\{(x, y) : 2 \leq 2x + 3y \leq 4\}$  is a closed set in  $\mathcal{R}^2$ .
5. Find  $\bar{S}$ , if  $S = \{(x, \sin(\frac{1}{x}))\}$ , for  $x \in \mathcal{R}^+$  and  $S$  in  $\mathcal{R}^2$ . Also find  $\text{int } S$  and  $\partial S$ .
6. The discrete topology on a space  $S$  is the one for which *every* subset is open. Show that if  $S$  is a finite metric space, then it has the discrete topology. (Hint show that there is an open ball about any point consisting entirely of that point. Then observe that unions of open sets are open).
7. Deduce that the topology on the space  $S = \{a, b, c\}$  with open sets  $\emptyset, S, \{a\}, \{a, b\}$  is not a metric space.