Math127- Solutions of metric space practice questions

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1 Show that if (M, d) and (P, d') are metric spaces, then so is the Cartesian product $M \times P$. Note that the metric on $M \times P$ can be defined in several different ways

- $D(x_1, x_2), (y_1, y_2) = d(x_1, x_2) + d'(y_1, y_2)$ or

-$D(x_1, y_1), (x_2, y_2) = \max\,\{d(x_1, x_2), d'(y_1, y_2)\}$

$D(x_1, y_1), (x_2, y_2) = \sqrt{(}d(x_1, x_2)^2 + d'(y_1, y_2)^2)$

Solution (sketchs)

We need to show that each of these is a metric, i.e symmetric, positive definite and satisfies the triangle inequality. The first two properties are easy in all cases to check and the triangle inequality is the tricky one. For the first and third definition , it is easy to check the triangle inequality. For the second, we need to show that max $\{d(x_1, x_3), d'(x_3, y_3)\} \leq$ max $\{d(x_1, x_2), d'(y_1, y_2)\}+$ max $\{d(x_2, x_3), d'(y_2, y_3)\}$. But since $d(x_1, x_3) \leq d(x_1, x_2)+d(x_2, x_3)$ and $d'(y_1, y_3) \leq d'(y_1, y_2) + d'(y_2, y_3)$, it is not too hard to see this follows.

2 In $\mathcal{R}^2$ show that the set $\{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ is closed. Show that the set $\{(x, y) : -1 < x < 1, -1 < y < 1\}$ is open. Finally prove that the boundary of either of these sets is the set $\{(x, y) : x = 1, -1 \leq y \leq 1; x = -1, -1 \leq y \leq 1; y = 1, -1 \leq x \leq 1; y = -1, -1 \leq x \leq 1\}$.

Solution

Sequences in $\mathcal{R}^2$ converge if and only if their coordihnates converge. So if we have a sequence of points (x_n, y_n) converges to a limit (x, y) and $-1 \leq x_n \leq 1, -1 \leq y_n \leq 1$, it follows that $-1 \leq x \leq 1, -1 \leq y \leq 1$. Hence $S = \{(x, y) : -1 \leq x \leq 1, -1 \leq y \leq 1\}$ contains all its limit points and is closed.

To show that $S = \{(x, y) : -1 < x < 1, -1 < y < 1\}$ is open, we can either show that for any point $(X, Y) \in S$, that there is a small open ball about the point contained in the set, or else the complement is closed, by the same argument as in the previous paragraph. To show the direct argument, if $(x, y) \in S$, then choose $\epsilon = min\{|1 - x|, |1 + x|, |1 - y|, |1 + y|\}$. It is easy to chcekc that the ball or radius ϵ centered at (x, y) is contained in S. (Draw a picture to see this!).

(Note a fancy way of showing these two facts is to define $f : (x, y) =$max $\{|1-x|, |1+x|, |1-y|, |1+y|\}$. Then one can show this is a continuous function and the first set is the inverse image of the closed set $[0, 1]$ and the second is the inverse image of the open set $(-1, 1)$.)

3 Show that if $f : M \to \mathcal{R}$ and $g : M \to \mathcal{R}$ are continuous functions then $f + g : M \to \mathcal{R}$ is continuous.

Solution

A direct solution would involve showing that if we have a sequence x_n in M converging to a limit x, then $(f+g)(x_n)$ converges to $(f+g)(x)$. But this is easy since $f(x_n)$ converges to $f(x)$ and $g(x_n)$ converges to $g(x)$ by continuity of f, g. So we just need to use the fact that sums of convergent sequences are convergent to the sum of the limits. This is also easy to prove directly.

4. Use a continuous function to show that $\{(x, y) : 2 \leq 2x + 3y \leq 4\}$ is a closed set.

Solution

Note that $f(x, y) = 2x + 3y$ is continuous from $\mathcal{R}^2$ to $\mathcal{R}$. So the inverse image of the closed set $[2, 4]$ is the closed set we want.

5. Find $\bar{S}$ if $S = \{(x, sin(\frac{1}{x}))\}$ for $x \in \mathcal{R}^+$. Also find int S and ∂S.

Solution

To find the closure $\bar{S}$, the easiest method is to find all the limit points, in particular the cluster points. Then by adding the cluster points to S we obtain the closure. Now it is not difficult to see that the cluster points of S are precisely the pairs $(0, y) : -1 \leq y \leq 1$, since we can find sequences of values of x converging to zero, which when put into the pair $(x, sin(\frac{1}{x}))$, we get convergence to a point $(0, y)$. On the other hand, any convergent sequence of points in the set must correspond to a convergent sequences of values of the first coordinate x. If the limit is not zero, by continuity of $sin(\frac{1}{x})$ for $x \neq 0$, then the limit point is already in the set. So it will not contribute a new point to the closure.

It is pretty easy to see that int S is empty, since there are no open disks inside S. Hence $\partial S = \bar{S}W$.

6. The discrete topology on a set S is the one for which every subset is open. Prove that if S is a finite metric space, then it has the discrete topology. (Hint: prove that there is an open ball about any point consisting of that point only. Then observe that the union of open sets is open)

Solution

If we have a finite metric space M, for any point $x \in M$, the minimum distance between x and all the other points of M is some positive number r. But then the open ball of radius r centered at x will only contain x, showing that $\{x\}$ is open. But then, an arbitrary set of points will be a union of open sets and hence open.

7.Deduce that the set $S = \{a, b, c\}$ with open sets $\emptyset, S, \{a\}, \{a, b\}$ is not a metric space.

Solution

The solution is easy - by 6 if this topololgy was a metric space, then all subsets would be open. But $\{b\}$ is not open,hence this is not a metric space.