UNIVERSITY OF MELBOURNE

DEPARTMENT OF MATHEMATICS AND STATISTICS

620-120 UMEP Maths for High Achieving Students

November 2005

SECOND PAPER

Exam duration: One and a half Hours

Reading Time: 15 Minutes

This paper consists of four (4) pages

Common content examinations Nil.

Authorised Materials
No materials are authorised.
Calculators and mathematical tables are not permitted.
Candidates are reminded that no written or printed material related to the subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators
No written or printed material related to the subject may be brought into the examination.
Students may take this paper with them from the examination room.

Instructions to Students
Marks for each question are shown on the paper.
The total number of marks available for this exam is 120.
All questions may be answered.
Students may take this paper with them from the examination room.

1. (a) (i) Define $\cosh(x)$ and $\sinh(x)$ in terms of exponentials.

(ii) Evaluate $\displaystyle\int_{-3}^{3} e^{tx}\,\mathrm{d}x$ giving your answer in the form $\alpha\sinh(3t)$ where α is a function of t.

(iii) Establish the identity $\cosh(2x) = 2\cosh^2(x) - 1$.

(iv) Find the area of the surface swept out by revolving about the x-axis the arc, C, of the curve

$$y = 4\cosh\left(\frac{x}{4}\right)$$

that lies between $x = -\log_e 16$ and $x = \log_e 81$.

(b) Establish the identity

$$\operatorname{arccosh}(x) = \log_e\left(x + \sqrt{x^2 - 1}\right), \quad \text{if } x \geq 1,$$

verifying the constraint on x.

[20 marks]

2. (a) Consider the equation

$$x^7 + 2x^3 = 2, \qquad x \in \mathbb{R}.$$

Show that this equation has exactly one root.
Quote fully any theorems that you use.

(b) (i) Use the Mean-value Theorem to show that

$$\sqrt{y} - \sqrt{x} < \frac{y - x}{2\sqrt{x}}, \quad \text{where } 0 < x < y.$$

(ii) **Hence** show that $\dfrac{12}{7} < \sqrt{3} < \dfrac{7}{4}$.

[20 marks]

3. (a) Consider the two closed polar curves, the rose $r = 2\cos(2\theta)$ and the circle $r = 2\sin(\theta)$.

(i) On the same diagram, sketch the two curves.

(ii) Find the points of intersection of the two curves.

(iii) Find the area of one 'petal' of the rose.

(b) Consider the curves $r_1 = f(\theta)$ and $r_2 = k\, f(\theta)$, where $\alpha \le \theta \le \beta$ and $k \in \mathbb{R} \setminus \{0\}$.

(i) Find the ratio of the lengths of the two curves.

(ii) The curves are rotated about the x-axis to generate surfaces. Find the ratio of the areas of the two surfaces.

[20 marks]

4. (a) By completing the square, show that

$$\int_0^{1/2} \frac{1}{x^2 - x + 1}\, \mathrm{d}x = \frac{\pi}{3\sqrt{3}}.$$

(b) (i) Write down the binomial series expansion for $(1+u)^k$, with k real and $|u| < 1$.

(ii) Using (i) above or otherwise, find the Maclaurin series expansion for $\dfrac{1}{1+x^3}$.

(iii) By writing $x^3 + 1$ in factor form, re-write the integral in (a).

(iv) Using your results in (ii) and (iii) show that

$$\pi = \frac{3\sqrt{3}}{4} \sum_{n=0}^{\infty} \left(\frac{-1}{8}\right)^n \left(\frac{2}{3n+1} + \frac{1}{3n+2}\right).$$

[20 marks]

5. (a) (i) Show that, for any integer $n \ge 2$,

$$\tan^n(x) = \frac{1}{n-1} \frac{\mathrm{d}}{\mathrm{d}x} \left\{\tan^{n-1}(x)\right\} - \tan^{n-2}(x).$$

(ii) Let $J_n = \displaystyle\int_0^{\pi/4} \tan^n(x)\, \mathrm{d}x$.

Show that

$$J_n = \frac{1}{n-1} - J_{n-2}.$$

(iii) Evaluate $\displaystyle\int_0^{\pi/4} \tan^8(x)\, \mathrm{d}x$.

(b) Given that

$$\int\int_R g(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,\mathrm{d}y\,\mathrm{d}x\,,$$

(i) sketch the region of integration R;

(ii) express the double integral in terms of a repeated integral in which integration with respect to x is performed first;

(iii) evaluate the integral.

[20 marks]

6. (a) Consider the function

$$f(x,y) = \frac{x}{y} + \frac{y}{x} - \frac{(x-y)^2}{a^2}\,.$$

(i) Find all stationary points of the function.

(ii) Discuss how the nature of the stationary points depends on the value of the parameter a.

(b) (i) Sketch the set of points in the complex plane representing the complex numbers, z, where

$$|z-1-2i| = 3\,.$$

(ii) If z satisfies $|z-1-2i| = 3$, find the set of points $u = z+4-3i$ in cartesian form. Describe this set geometrically.

(iii) Find the greatest and least values of $|z-4-6i|$ if z is subject to the inequality $|z-1-2i| \leq 3$.

[20 marks]

END OF EXAMINATION