UNIVERSITY OF MELBOURNE

DEPARTMENT OF MATHEMATICS AND STATISTICS

620-120 UMEP Maths for High Achieving Students

November 2006

SECOND PAPER

Exam duration: Two Hours

Reading Time: 15 Minutes

This paper consists of four (4) pages

Common content examinations Nil.

Authorised Materials
No materials are authorised.
Calculators and mathematical tables are not permitted.
Candidates are reminded that no written or printed material related to the subject may be brought into the examination. If you have any such material in your possession, you should immediately surrender it to an invigilator.

Instructions to Invigilators
No written or printed material related to the subject may be brought into the examination.
Students may take this paper with them from the examination room.

Instructions to Students
Marks for each question are shown on the paper.
The total number of marks available for this exam is 120.
All questions may be answered.
Students may take this paper with them from the examination room.

1. (a) (i) Define $\sinh(x)$ and $\cosh(x)$ in terms of exponentials.

(ii) Show that $\sinh^2(x)\cos^2(y) + \cosh^2(x)\sin^2(y) = \sinh^2(x) + \sin^2(y)$.

(b) (i) Establish the identity

$$\operatorname{arcsinh}(x) = \log_e(x + \sqrt{1+x^2}).$$

(ii) Write down the binomial series expansion for $(1+u)^k$, with k real and $|u| < 1$.

(iii) Using (ii) or otherwise, find the Maclaurin series for $\dfrac{1}{\sqrt{1+u^2}}$.

(iv) **Hence**, find the degree 5 Taylor polynomial for $\log_e(x + \sqrt{1+x^2})$ expanded about $x = 0$.

[20 marks]

2. (a) Prove carefully that the polynomial

$$p(x) = 2x^3 + 3x^2 - 12x + 1$$

attains the value 9 exactly three times. State any theorems that you use.

(b) The Mean Value Theorem states that for a sufficiently well behaved function f defined on the interval $[a, b]$,

$$f(b) - f(a) = f'(c)(b-a) \quad \text{for some } c \text{ with } a < c < b.$$

(i) State, but do not prove, sufficient conditions on the function f for the Mean Value Theorem to hold.

(ii) Use the Mean Value Theorem to prove that for all $x \neq 0$,

$$\frac{\sinh x}{x} > 1\,.$$

(iii) Use the Mean Value Theorem to prove that if a function f has a derivative which is zero at all points of an open interval, then f is equal to a constant on that interval.

[20 marks]

3. (a) (i) Let f be a bounded, continuous function on the interval $[a, b]$ such that $f(x) > 0$ for all x in $[a, b]$. Write down the Riemann sum for the area under f using n subintervals with right hand end-points.

(ii) Give the boundaries of the region whose area is approximated by the Riemann sum

$$S_n = \sum_{k=1}^{n} \frac{2n}{k^2 + n^2} .$$

(iii) **Hence** find $\lim_{n \to \infty} S_n$.

(b) Let $I = \int_1^3 3^x \, \mathrm{d}x$.

(i) Calculate the Simpson approximation for I using two subintervals.

(ii) Show that $I = \dfrac{24}{\log_e 3}$.

(iii) For any function f with continuous fourth derivative on the interval $[a, b]$, the Simpson two-interval approximation I_S for $\int_a^b f(x) \, \mathrm{d}x$ is such that there is some c : $a < c < b$ for which

$$\int_a^b f(x) \, \mathrm{d}x = I_S - \frac{(b-a)^5}{90} f^{(iv)}(c) .$$

Use this result to show that $\log_e 3 > \dfrac{12}{11}$.

[20 marks]

4. Consider the two closed polar curves

$$r = 1 - \cos\theta \quad \text{and} \quad r = 1 + \cos\theta , \quad 0 \le \theta \le 2\pi.$$

(i) Sketch the polar curves, clearly labeling the coordinates of all points of intersection.

(ii) Find the area of the region bounded by the polar curves.

(iii) The region bounded by the polar curves is rotated about the x-axis. Calculate the surface area of the solid of revolution formed.

(iv) The points on the polar curves that have horizontal tangents all lie on a circle centred at the origin. Determine the polar equation for this circle.

[20 marks]

5. Consider the integrals

$$I = \int_0^{\pi/2} \sin(3x)\sinh(4x)\,\mathrm{d}x \quad \text{and } J = \int_0^{\pi/2} \cos(3x)\cosh(4x)\,\mathrm{d}x$$

(a) Use integration by parts in the definition of the integral I to relate it to J. Also use integration by parts in the definition of the integral J to relate it to I. **Hence** evaluate I and J without performing any further integration.

(b) Evaluate the integrals

$$\int_0^{\pi/2} e^{(4+3i)x}\,\mathrm{d}x \quad \text{and} \quad \int_0^{\pi/2} e^{(-4+3i)x}\,\mathrm{d}x\,.$$

Hence evaluate I and J and confirm your answers to part (a).

[20 marks]

6. (a) Consider the function

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2.$$

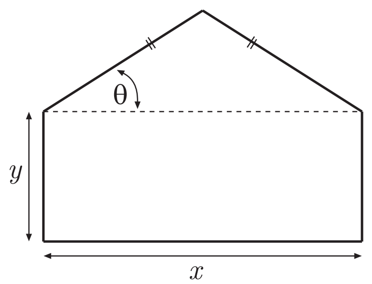
(i) Find any local maximum and minimum value(s) of f.

(ii) Now consider f on the triangular region R in the first quadrant bounded by the lines $x = 0$, $y = 0$ and $y = 9 - x$. Find the maximum and minimum values of f on the boundary of R.

(iii) Give the absolute maximum and minimum values of f on the region R and the points at which they occur.

(b) Show that the sum of the x, y and z intercepts of any tangent plane to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{d}$ is a constant.

(c) A window has the shape of a rectangle surmounted by an isosceles triangle, as shown in the diagram below. If the perimeter of the window is 6 metres, what values of x, y and θ will maximize the total area?



[20 marks]

END OF EXAMINATION QUESTIONS