Solutions to 620-142 Mathematics B Exam
Semester One 2006

1. (a)

$$\begin{aligned} 425 &= 1 \times 357 + 68 \\ 357 &= 5 \times 68 + \mathbf{17} \\ 68 &= 4 \times 17 + 0 \end{aligned}$$

   Thus $\gcd(425, 357) = 17$.

   (b)

$$\begin{aligned} 3x &= 4^3 \\ \Rightarrow 3x &= (-1)^3 \\ \Rightarrow 2 \times 3x &= 2 \times -1 \\ x &= 3. \end{aligned}$$

   (c)

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \end{array}\right]$$

   Let $z = t \in \mathbb{Z}_3$ so $x = 1 - 2t = 1 + t$ and $y = 2 - t = 2 + 2t$ thus enumerating completely ($t = 0, t = 1, t = 2$) we obtain (1,2,0), (2,1,1), (0,0,2) as the complete set of solutions.

2. (a) $n = \phi(m) = \phi(91) = \phi(7 \times 13) = (7-1)(13-1) = 6 \times 12 = 72$.

   (b) Seek d so that $ed \equiv 1 \mod \phi(m)$, that is, find a multiplicative inverse to $e = 5$ mod 72.

$$\begin{aligned} 72 &= 14 \times 5 + 2 \\ 5 &= 2 \times 2 + 1 \qquad gcd \\ 2 &= 2 \times 1 + 0 \end{aligned}$$

$$\begin{aligned} 1 &= 5 - 2 \times 2 \\ &= 5 - 2 \times (72 - 14 \times 5) \\ &= 29 \times 5 - 2 \times 72 \end{aligned}$$

   $29 \times 5 = 1 + 2 \times 72 \equiv 1 \mod 72$; $d = 29$ is inverse to $e = 5, \mod 72$.

   (c) $53^{29} = ?$: Using binary powering:

$$\begin{array}{ccc} 53 & 29 & * \\ 79 & 14 & \\ 53 & 7 & * \\ 79 & 3 & * \\ 53 & 1 & * \end{array}$$

So

$$\begin{aligned} 53^{29} &= 53 \times (53 \times 53) \times 79 \\ &= 53 \times (79 \times 79) \\ &= 53 \times 53 \\ &= 79 \end{aligned}$$

Decrypt the 'message' 53 as 79.

3. (a) $6^2 = 36 = 36 - 2 \times 17 = 2$.
   (b) 17 is a *prime* so there are $\phi(17) = 17 - 1 = 16$ multiplicative units in $\mathbb{Z}_{17}$.
   (c) *Fermat's Little Theorem* says that $x^{16} = 1$ for all $x \neq 0 \in \mathbb{Z}_{17}$ thus the order of a unit divides 16 and so is in $\{1, 2, 4, 8, 16\}$.
   (d) The order does not divide 4 (as $2^4 = -1 \neq 1$) but $2^8 = (2^4)^2 = (-1)^2 = 1$ so the order divides 8, and thus must be 8.
   (e) * $6^2 = 2$ so as all orders in $\mathbb{Z}_{17}$ are powers of 2, the order of 6 is twice the order of 2 and thus is 16.

4. (a) $\text{rank}A = \text{rank}B = 3 =$ number of non-zero rows in row-echelon form.
   (b) $\{(2, -4, 2, 3), (5, -10, 1, 7), (0, 0, -1, 8)\}$
   (i.e., the first, second and fifth columns of A)
   (c) $\{(1, 0, 3, 4, 0), (0, 1, -2, 6, 0), (0, 0, 0, 0, 1)\}$: non-zero rows of B.
   (d) No. There are four rows, but their span has dimension 3.
   (e) No. The column space has dimension 3, and is therefore not the whole of $\mathbb{R}^4$.
   (f) $(-4, 8, 4, -5) = 3(2, -4, 2, 3) - 2(5, -10, 1, 7)$ for columns in A (see column 3 of B).
   (g) We have
   $$\text{rank}(T) + \text{nullity}(T) = \dim(\mathbb{R}^5) = 5;$$
   $\text{rank}(T) = 3$, so $\text{nullity}(T) = 2$.

5. (a) $(0, 1)$ and $(1, 0) \in V = \{(x, y) \in \mathbb{R}^2 : x \times y = 0\}$ (the *union* of the x and y axes) BUT $(1, 1) = (0, 1) + (1, 0) \notin V$ as $1 \times 1 \neq 0$.
   (b) Let W be the span of the set
   $$\mathcal{B} = \{(2, 2, 1, 0), (-2, 2, 0, 1)\}.$$
   (i) An *orthonormal basis* is $\{\boldsymbol{u}_1, \boldsymbol{u}_2\}$ where,
   $$\begin{aligned} \boldsymbol{u}_1 &= \frac{1}{\|\boldsymbol{b}_1\|}\boldsymbol{b}_1 = \frac{1}{3}(2, 2, 1, 0) \\ \boldsymbol{w}_2 &= \boldsymbol{b}_2 - (\boldsymbol{b}_2 \cdot \boldsymbol{u}_1)\boldsymbol{u}_1 \\ &= \boldsymbol{b}_2 - 0 \times \boldsymbol{u}_1 = \boldsymbol{b}_2 \\ \boldsymbol{u}_2 &= \frac{1}{\|\boldsymbol{w}_2\|}\boldsymbol{w}_2 = \frac{1}{3}(-2, 2, 0, 1) \end{aligned}$$

(ii)

$$\begin{aligned}
\boldsymbol{p} = \mathrm{Proj}_W(\boldsymbol{v}) &= (\boldsymbol{v}\cdot\boldsymbol{u}_1)\,\boldsymbol{u}_1 + (\boldsymbol{v}\cdot\boldsymbol{u}_2)\,\boldsymbol{u}_2 \\
&= \left((3,6,0,3)\cdot\frac{1}{3}(2,2,1,0)\right)\frac{1}{3}(2,2,1,0) \\
&\quad + \left((3,6,0,3)\cdot\frac{1}{3}(-2,2,0,1)\right)\frac{1}{3}(-2,2,0,1) \\
&= 2(2,2,1,0) + 1(-2,2,0,1) = (2,6,2,1)
\end{aligned}$$

(iii)

$$\begin{aligned}
\boldsymbol{w}_3 &= \boldsymbol{v} - \mathrm{Proj}_{\langle \boldsymbol{u}_1,\boldsymbol{u}_2\rangle}(\boldsymbol{v}) \\
&= (3,6,0,3) - (2,6,2,1) = (1,0,-2,2) \\
\boldsymbol{u}_3 &= \frac{1}{\|\boldsymbol{w}_3\|}\boldsymbol{w}_3 = \frac{1}{3}(1,0,-2,2)
\end{aligned}$$

So an *orthonormal basis* for W' is

$$\left\{\frac{1}{3}(2,2,1,0), \frac{1}{3}(-2,2,0,1), \frac{1}{3}(1,0,-2,2)\right\}.$$

6. (a) Using syndromes we see that the first and third are codewords (the second and fourth have non-zero syndromes).

(b) B is in row-echelon form: rank 7, nullity $9-7=2=$ dimension of solution space.

(c) There are $2^{\mathrm{dim}} = 2^2 = 4$ code words.

(d) The two codewords above are independent since they are distinct. They give a basis since the subspace of codewords is 2-dimensional.

(e) Codewords are $\{000000000, 101110101, 011101110, 110011011 = 101110101 + 011101110\}$. The middle two are independent (in fact any non-zero pair is a basis).

(f) The Hamming distance is the least number of 1's in a non-zero code word for linear codes: this is 6, the same for all words in this code.

(g) This code will detect $6-1=5$ errors, and will correct 2 errors (since $2\times 2+1 = 5 \le 6$).

(h) The first satisfies

$$d(000000001, 000000000) = 1 < 2,\ d(000000001, 101110101) = 5 > 2,$$

$$d(000000001, 011101110) = 7 > 2,\ d(000000001, 110011011) = 5 > 2$$

The corrected word is thus 000000000 (its nearest neighbour).
The second is already a code word, and is correct;
the third has

$$d(111101111, 000000000) = 8,\ d(111101111, 101110101) = 4 > 2,$$

$$d(111101111, 011101110) = 2,\ d(111101111, 110011011) = 4 > 2$$

so the nearest neighbour is 011101110.

(i) It cannot correct a word with three errors such as $0***01110$ with $*** = 000$. This word 000001110 is half way between 000000000 and 011101110. Moreover

$$d(000001110, 101110101) = 7,\ d(000001110, 110011011) = 5.$$

7. (a)

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(b) There are 6 paths with 3 edges from 1 to 2.

(c) There are $[1\ 1\ 1\ 1][4\ 2\ 2\ 1]^T = 9$ paths with 5 edges from 4 to 4.

8. (a)

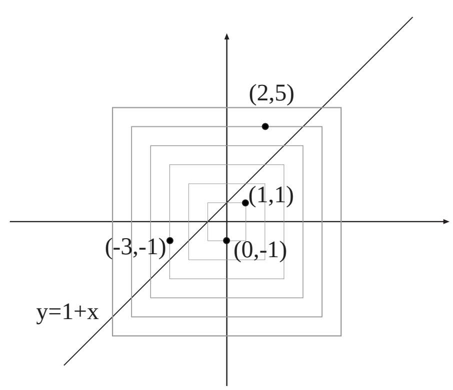
$$A = \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \boldsymbol{y} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 5 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 14 \end{bmatrix}$$

$$(A^T A)^{-1} = \begin{bmatrix} 4^{-1} & 0 \\ 0 & 14^{-1} \end{bmatrix}, \qquad A^T \boldsymbol{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$$

$$(A^T A)^{-1} A^T \boldsymbol{y} = \begin{bmatrix} 4^{-1} & 0 \\ 0 & 14^{-1} \end{bmatrix} \begin{bmatrix} 4 \\ 14 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So the line of best fit is $y = 1 + x$.

(b)



(c)

$$A = \begin{bmatrix} 1 & -3 & 9 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$$

9. (a) (i) $\langle (u_1,u_2),(v_1,v_2)\rangle = u_1v_1 + u_1v_2$;
The formula is not symmetric (the coefficient of u_1v_2 is 1 but the coefficient of v_1u_2 is 0). An explicit counter example to symmetry is $\langle ((1,0),(0,1)\rangle = 1$ but $\langle ((0,1),(1,0)\rangle = 0$.

(ii) * $\langle (u_1,u_2),(v_1,v_2)\rangle = 3u_1v_1 - u_1v_2 - u_2v_1$.
The formula is not positive definite $(0,1) \neq \mathbf{0}$ but $\langle (0,1),(0,1)\rangle = 0$.

(b) (i)

$$\langle \boldsymbol{a},\ \boldsymbol{b}\rangle = \boldsymbol{a}^T A \boldsymbol{b} \text{ where } A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{bmatrix},$$

or equivalently, $\langle (a_1,a_2),(b_1,b_2)\rangle = 2a_1b_1 + \frac{2}{3}a_2b_2$.

Checking the inner product *axioms.*

symmetry

$$\begin{aligned} \langle (a_1,a_2),(b_1,b_2)\rangle &= 2a_1b_1 + \frac{2}{3}a_2b_2 \\ &= 2b_1a_1 + \frac{2}{3}b_2a_2 \\ &= \langle (b_1,b_2),(a_1,a_2)\rangle \end{aligned}$$

linearity

$$\begin{aligned} \langle \boldsymbol{a},\boldsymbol{b}+\boldsymbol{c}\rangle &= \boldsymbol{a}^T A(\boldsymbol{b}+\boldsymbol{c}) \\ &= (\boldsymbol{a}^T A)\boldsymbol{b} + (\boldsymbol{a}^T A)\boldsymbol{c} \\ &= \langle \boldsymbol{a},\boldsymbol{b}\rangle + \langle (\boldsymbol{a},\boldsymbol{c}\rangle \end{aligned}$$

$$\begin{aligned} \langle \boldsymbol{a},\beta\boldsymbol{b}\rangle &= \boldsymbol{a}^T A(\beta\boldsymbol{b}) \\ &= \boldsymbol{a}^T \beta A\boldsymbol{b} \\ &= \beta\boldsymbol{a}^T A\boldsymbol{b} \\ &= \beta\,\langle \boldsymbol{a},\boldsymbol{b}\rangle \end{aligned}$$

positivity

$$\langle (a_1,a_2),(a_1,a_2)\rangle = 2a_1^2 + \frac{2}{3}a_2^2 \geq 0 \text{ for all } (a_1,a_2) \in \mathbb{R}^2.$$

definiteness From above

$$\langle (a_1,a_2),(a_1,a_2)\rangle = 0 \Leftrightarrow a_1^2 = 0 \ \&\&\ a_2^2 = 0 \Leftrightarrow (a_1,a_2) = \mathbf{0} \in \mathbb{R}^2.$$

(ii)

$$\|(1,2)\| = (\langle (1,2),(1,2)\rangle)^{\frac{1}{2}} = \left(2\times 1^2 + \frac{2}{3}\times 2^2\right)^{\frac{1}{2}} = \sqrt{\frac{14}{3}}$$

$$\langle (1,2),(2,-3)\rangle = [1\ \ 2]\begin{bmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = [2\ \ \frac{4}{3}]\begin{bmatrix} 2 \\ -3 \end{bmatrix} = 0.$$

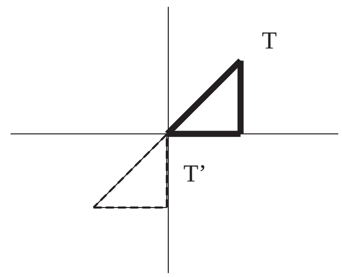
10\. (a) (i)

$$A_S = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(ii)

$$A_{R\circ S} = A_R A_S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

(iii)



(b) Consider the *symmetric* matrix

$$A = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$

(i) $\det(A - \lambda I) = \lambda^2 - (\frac{4}{5})^2 - (\frac{3}{5})^2 = \lambda^2 - 1$ so the *eigenvalues* of A are $-1,\ 1$.

(ii) Find the corresponding *eigenvectors* of A.

$\lambda = 1$

$$\begin{bmatrix} -\frac{4}{5} - 1 & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} - 1 \end{bmatrix} \sim \begin{bmatrix} -9 & 3 \\ 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix}$$

by inspection the *eigenspace* has dimension 1 and is spanned by $[1\ \ 3]^T$. Thus any non-zero multiple of $[1\ \ 3]^T$ is an eigenvector for eigenvalue 1.

$\lambda = -1$

$$\begin{bmatrix} -\frac{4}{5} - (-1) & \frac{3}{5} \\ \frac{3}{5} & \frac{4}{5} - (-1) \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix}$$

by inspection the *eigenspace* has dimension 1 and is spanned by $[3\ \ -1]^T$ (which is perpendicular to the other eigenvector as must be the case for symmetric matrices). Thus any non-zero multiple of $[-3\ \ 1]^T$ is an eigenvector for eigenvalue -1.

(iii) * As described above the eigenvectors are perpendicular.

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \xrightarrow{T} \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ -1 \end{bmatrix} \xrightarrow{T} -1 \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Thus T is a reflection about the line (through $\mathbf{0}$) parallel to the vector $[1\ \ 3]^T$ that is a reflection about $y = 3x$.

11. (a)

$$X^t A X := \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -2 & -8 & -2 \\ -8 & 4 & 10 \\ -2 & 10 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 1$$

(b) These are eigenvectors of A with eigenvalues 18, 0, −9:

$$A \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = 18 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \qquad A \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \qquad A \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = -9 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix};$$

(c) Since A is symmetric, it has orthogonal eigenspaces. Since eigenvalues are distinct, these eigenspaces are 1-dimensional and orthogonal. A matrix P with columns the three eigenvectors diagonalizes A:

$$P = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}, \qquad D = P^{-1}AP = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

(d) Normalize the basis of eigenvectors: each has length $\sqrt{9} = 3$:

$$Q = \frac{1}{3}P = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

(e) $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^{\mathbf{T}}\mathbf{A}\mathbf{v}$ is not an inner product on $\mathbb{R}^3$ since the matrix does not have all eigenvalues > 0. For example,

$$\langle (2, 2, -1), (2, 2, -1) \rangle = -81 < 0.$$

(f) The quadratic form $F : \mathbb{R}^3 \to \mathbb{R}$ defined by $F(\mathbf{x}) = \mathbf{x}^{\mathbf{T}}\mathbf{A}\mathbf{x}$, is equivalent to $18a^2 + 0.b^2 + (-9)c^2$, hence to $u^2 - w^2$,

12. (a)

$$[T]_{\mathcal{S}\to\mathcal{S}} = \begin{bmatrix} 8 & -3 \\ 18 & -7 \end{bmatrix}.$$

(b)

$$P_{\mathcal{B}\to\mathcal{S}} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}.$$

(c)

$$P_{\mathcal{S}\to\mathcal{B}} = (P_{\mathcal{B}\to\mathcal{S}})^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}.$$

(d)

$$\begin{aligned} [T]_{\mathcal{B}\to\mathcal{B}} &= P_{\mathcal{S}\to\mathcal{B}}[T]_{\mathcal{S}\to\mathcal{S}}P_{\mathcal{B}\to\mathcal{S}} \\ &= \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 8 & -3 \\ 18 & -7 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

(e)

$$[\boldsymbol{w}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \text{ and } [T(\boldsymbol{w})]_{\mathcal{B}} = [T]_{\mathcal{B}\to\mathcal{B}}\,[\boldsymbol{w}]_{\mathcal{B}} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

13. (a) T is a linear transformation:

$$T(B+C) = A(B+C) - (B+C)A = A(B) - (B)A + A(C) - (C)A = T(B) + T(C)$$

$$T(\alpha B) = A(\alpha B) - (\alpha B)A = \alpha(AB - BA) = \alpha T(B).$$

(b)

$$T(E_1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = -E_2 - E_3$$

$$T(E_2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = E_1 - E_4$$

$$T(E_3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = E_1 - E_4$$

$$T(E_4) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_2 + E_3$$

(c) The standard matrix $[T]_{\mathcal{S}\to\mathcal{S}}$ of T is

$$[T]_{\mathcal{S}\to\mathcal{S}} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

(d) Row reduction gives

$$[T]_{\mathcal{S}\to\mathcal{S}} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

The image is the span of $(E_1 - E_4), (-E_2 - E_3)$, corresponding to the first two columns of $[T]_{\mathcal{S}\to\mathcal{S}}$, that is

$$\left\{ \begin{bmatrix} s & -t \\ -t & -s \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

The kernel (from the solution space of $[T]_{\mathcal{S}\to\mathcal{S}}$) is the set

$$\{s(E_1 + E_4) + t(E_2 - E_3) \ : \ s, t \in \mathbb{R}\} = \begin{bmatrix} s & t \\ -t & s \end{bmatrix}.$$

$$= \{a_1E_1 + a_2E_2 + a_3E_3 + a_4E_4 \ : \ a_1 = a_4 = s, \ a_2 = -a_3 = t\}$$

(e) The dimensions of the kernel and image of T are both 2:

$$\dim \operatorname{Ker}(T) + \dim \operatorname{Im}(T) = \dim(M_{2\times 2}(\mathbb{R}); \qquad 2 + 2 = 4.$$

(f) Since $T(B) = AB - BA$, we find $T(A) = AA - AA = 0$. Hence if $A \neq 0$, T has a non-zero kernel, and hence T must have image with dimension less than 4.