

1. *Solution.*

(a)

$$\begin{aligned}1033 &= 10 \times 100 + 33 \\100 &= 3 \times 33 + \underline{1}\end{aligned}$$

(b)

$$\begin{aligned}1 &= 100 - 3 \times 33 \\&= 100 - 3 \times (1033 - 10 \times 100) \\&= 31 \times 100 - 3 \times 1033\end{aligned}$$

Thus in \mathbb{Z}_{1033} , $1 = 31 \times 100 - 3 \times 1033 = 31 \times 100$ meaning $100^{-1} = 31$ in \mathbb{Z}_{1033} .

(c)

$$\begin{aligned}100x &= -3 \\ \Rightarrow 31 \times 100x &= 31 \times -3 \\ \Rightarrow x &= -93 \\ &= 940\end{aligned}$$

(d) The inverse of 15 in \mathbb{Z}_{85} does not exist as $\gcd(15, 85) = 5 \neq 1$.

2. *Solution.*

Let \mathcal{S}_n be the proposition that $A^n \mathbf{u} = \lambda^n \mathbf{u}$ where $n \in \mathbb{Z}^+$.

Base case

\mathcal{S}_1 : $A^1 \mathbf{u} = \lambda^1 \mathbf{u}$ as \mathbf{u} is an eigenvector of A with eigenvalue λ ($A\mathbf{u} = \lambda\mathbf{u}$).

Inductive step

Assume \mathcal{S}_k is true (where $k \in \mathbb{Z}^+$), that is

$$\begin{aligned} A^k \mathbf{u} &= \lambda^k \mathbf{u} \\ \text{thus } A^{k+1} \mathbf{u} &= A(A^k \mathbf{u}) \\ &= A(\lambda^k \mathbf{u}) \quad (\text{by } \mathcal{S}_k) \\ &= \lambda^k (A\mathbf{u}) \\ &= \lambda^k \lambda \mathbf{u} \\ &= \lambda^{k+1} \mathbf{u} \end{aligned}$$

this is \mathcal{S}_{k+1} .

So $\mathcal{S}_k \Rightarrow \mathcal{S}_{k+1}$.

So by the principle of mathematical induction \mathcal{S}_n : $A^n \mathbf{u} = \lambda^n \mathbf{u}$ for any positive integer n .

3. *Solution.*

- (a) $n = \phi(m) = \phi(3 \times 29) = (3 - 1) \times (29 - 1) = 2 \times 28 = 56$.
 (b) Encrypting the message '16' is done by 16^{11} in \mathbb{Z}_{87} .

16	11
82	5
25	2
16	1

So (looking at the odd entries in the RH column) $16^{11} = 16 \times 82 \times 16 = 16^2 \times 82 = 82 \times 82 = 82^2 = 25$. Thus '16' is encrypted as '25'.

- (c) The decrypting key d needs to be the inverse of $e \pmod n$ (NOT $\pmod m$)! $11 \times 51 = 561 = 561 - 10 \times 56 = 1$ in \mathbb{Z}_{56} . Whereas $11 \times 8 = 88 = 32 \neq 1$ in \mathbb{Z}_{56} . Thus the appropriate decrypting key is $d = 51$.
 (d) We need $62^{51} = (-25)^{51} = -(25^{51})$.

25	51
16	25
82	12
25	6
16	3
82	1

So (looking at the odd entries in the RH column) $25^{51} = 25 \times 16 \times 16 \times 82 = 25 \times 16^2 \times 82 = 25 \times 82 \times 82 = 25 \times 82^2 = 25 \times 25 = 25^2 = 16$, so $-(25^{51}) = -16 = 71$ in \mathbb{Z}_{87} . Thus '62' is decrypted as '71'.

4. *Solution.* Current algorithms and computing power mean that prime-factorising a 400 digit m to find p and q may well take more than a lifetime. Note that we need p and q to find $n = (p - 1)(q - 1)$ to in turn find $d = e^{-1}$ in \mathbb{Z}_n .

5. *Solution.*

- (a) As $3^6 = 1$ the order of 3 divides 6, thus the possibilities are 1,2,3 or 6. Now $3^1 \neq 1$, $3^2 = 9 \neq 1$ but $3^3 = 27 = 1$. So the order of 3 is 3.
 (b) Now $\phi(26) = \phi(2 \times 13) = (2 - 1) \times (13 - 1) = 12$, so (by Euler's Theorem) the order of any unit divides 12, giving 1, 2, 3, 4, 6, 12 as the possibilities.
 Now $15^6 = 3^6 \times 5^6 = (3^3)^2 \times 5^4 \times 5^2 = 1 \times 1 \times -1 \neq 1$. Thus the order of 15 is not 6 (or anything dividing 6). Also $15^4 = 3^4 \times 5^4 = 3^3 \times 3^1 \times 1 = 1 \times 3 = 3 \neq 1$ so the order of 15 is not 4 (or anything dividing 4). This leaves $\cancel{1}, \cancel{2}, \cancel{3}, \cancel{4}, \cancel{6}, 12$ as the only possibility. As the order of 15 in \mathbb{Z}_{26} is $\phi(26)$ it is a primitive unit.
 (c) $13 \cong 1 \pmod{\phi(26)}$ so by the extended Eulers Theorem $a^{13} = a$ in \mathbb{Z}_{26} for any a .

- (a) The rank of A is 3.
 (b) The first, second and fourth columns of the original matrix,

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 3 \\ -7 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

- (c) The dimension of the row space of A is the rank of A which is 3.
 (d) The rows of A are not linearly independent – it is impossible for 4 vectors to be linearly independent in a space of dimension < 4 .
 (e) The non-trivial rows of B form a basis for the row space of A ,
 $\{(1, 0, 1, 0, 1), (0, 1, -1, 0, 0), (0, 0, 0, 1, 4)\}$.
 (f) No the vectors $(1, -2, 1, 2), (-2, 4, 3, -7), (3, -6, -2, 9), (1, -2, 1, 0), (5, -10, 5, 2)$ do not span \mathbb{R}^4 – these vectors are the columns of A and their span (the column space) is only of dimension 3. Give a reason.
 (g) $(3, -6, -2, 9) = 1(1, -2, 1, 2) - 1(-2, 4, 3, -7)$ – see column 3 of B .
 (h) Let $x_3 = s$ and $x_5 = t$ (the third and fifth columns of B have no row leaders). Row 3 $\Rightarrow x_4 = -4t$. Row 2 $\Rightarrow x_2 = s$. Row 1 $\Rightarrow x_1 + s + t = 0 \Rightarrow x_1 = -s - t$. Thus to be in the solution space $(x_1, x_2, x_3, x_4, x_5) = (-s - t, s, s, -4t, t) = s(-1, 1, 1, 0, 0) + t(-1, 0, 0, -4, 1)$. So a basis for the solution space is

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}.$$

7. (i) *Solution.* The kernel of T is the solution space of A thus a basis is the one given to the previous question part.

- (a) Now the vector $\mathbf{w} = (-1, -1, -1)$ is in P as $(-1) + (-1) + (-1) \leq 0$. But $-1\mathbf{w} = (1, 1, 1)$ is not in P as $1 + 1 + 1 \not\leq 0$! So P isn't closed under scalar multiplication and is not a subspace of \mathbb{R}^3 .
 (b) Let

$$T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$$

be a linear transformation.

Suppose that $T(\mathbf{u}) = \mathbf{u}'$, $T(\mathbf{v}) = \mathbf{v}'$.

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{u}' + \mathbf{v}'$ and $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u}) = \alpha\mathbf{u}'$.

(ii)

$$\text{Ker}(T) = \{\mathbf{w} \in \mathbb{R}^5 : T(\mathbf{w}) = \mathbf{0} \in \mathbb{R}^4\}.$$

(iii) $\text{Ker}(T)$ contains $\mathbf{0} \in \mathbb{R}^5$ as $T(\mathbf{0}) = T(0\mathbf{w}) = 0T(\mathbf{w}) = \mathbf{0} \in \mathbb{R}^4$. So $\text{Ker}(T)$ is non-empty.

Suppose both \mathbf{u} and \mathbf{v} are in $\text{Ker}(T)$ then $T(\mathbf{u}) = \mathbf{0}$ and $T(\mathbf{v}) = \mathbf{0}$ so (with $\mathbf{u}' = \mathbf{0}$ and $\mathbf{v}' = \mathbf{0}$) we see $T(\mathbf{u} + \mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}$. Thus $\mathbf{u} + \mathbf{v}$ is in $\text{Ker}(T)$.

Thus $\text{Ker}(T)$ is closed under addition.

$T(\alpha\mathbf{v}) = \alpha T(\mathbf{v}) = \alpha\mathbf{0} = \mathbf{0}$ so $\alpha\mathbf{v}$ is in $\text{Ker}(T)$ and $\text{Ker}(T)$ is closed under scalar multiplication.

Thus by the subspace Theorem $\text{Ker}(T)$ is a subspace of \mathbb{R}^5 .

8. *Solution.*

- (a) The word 0001111 is a codeword as

$$B[0\ 0\ 0\ 0\ 1\ 1\ 1]^T = [0\ 0\ 0\ 0]^T$$

whereas the other two words are not codewords as they have non-zero syndrome.

- (b) The dimension of \mathcal{C} is 3 so there are $|\mathbb{Z}_2|^3 = 2^3 = 8$ codewords.
- (c) The minimum distance of the code is 4 so the code can:
- (i) correct 1 error (but no more than 1);
 - (ii) detect 3 errors.
- (d) 1000000 \rightarrow 0000000 (the zero word is always a codeword), 0001011 has syndrome 0111 which is the 5th column of the check matrix (so the single error is in the 5th bit) thus 0001011 \rightarrow 0001111. (0001111 is a codeword).
- (e) The rank of the matrix B is 4 as it is already in RREF form. Thus by the rank/nullity theorem the dimension of the solution space is (the number of columns - the rank) $7-4 = 3$. The solution space of B is exactly the code \mathcal{C} thus \mathcal{C} has dimension 3.
- (f) The word 1100000 has at least 2 errors as the syndrome 1100^T is Not 0000^T (hence at least one error is made) and is NOT a column of the check matrix – if only one error is made in the i th bit then the syndrome is B_i the i th column of the check matrix B .

9. Solution.

- (a) (i) With the data points $\frac{\mathbf{x}}{\mathbf{y}} \left| \begin{array}{ccccc} -4 & -2 & 1 & 2 & 3 \\ 5 & 5 & 2 & 0 & -2 \end{array} \right.$ we have $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 1 & 2 & 3 \end{bmatrix}^T$.
So

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & -2 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 34 \end{bmatrix},$$

$$A^T \mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 2 \\ 0 \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} 10 \\ -34 \end{bmatrix}$$

Solving

$$A^T A \bar{\mathbf{u}} = A^T \mathbf{y} \text{ for } \bar{\mathbf{u}} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\text{gives } \begin{array}{rcl} 5a & = & 10 \\ 34b & = & -34 \end{array}$$

$$\text{So } a = 2, b = -1.$$

So the line of (least squares) best fit is $y = 2 - x$.

- (ii) The points on the line are $\frac{\mathbf{x}}{\mathbf{y}'} \left| \begin{array}{ccccc} -4 & -2 & 1 & 2 & 3 \\ 6 & 4 & 1 & 0 & -1 \end{array} \right.$ giving a least squares error of $(6 - 5)^2 + (4 - 5)^2 + (1 - 2)^2 + (0 - 0)^2 + (-1 - -2)^2 = 4$.

(b)

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4 & -2 & 1 & 2 & 3 \\ 16 & 4 & 1 & 4 & 9 \end{bmatrix}^T$$

10. Solution.

- (a) (i) The properties $\langle \cdot, \cdot \rangle$ must satisfy (to be an inner product on \mathbb{R}^2) are:
- 1 $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all \mathbf{u}, \mathbf{v} in \mathbb{R}^2 .
 - 2 $\langle \mathbf{u}, \alpha \mathbf{v} \rangle = \alpha \langle \mathbf{u}, \mathbf{v} \rangle$ for all $\alpha \in \mathbb{R}$ and for all \mathbf{u}, \mathbf{v} in \mathbb{R}^2 .
 - 3 $\langle \mathbf{u}, (\mathbf{v} + \mathbf{w}) \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in \mathbb{R}^2 .
 - 4a $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for all \mathbf{u} in \mathbb{R}^2 .
 - 4b $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$ in \mathbb{R}^2 .

(ii) $\|(3, 2)\| = (\langle(3, 2), (3, 2)\rangle)^{1/2} = (3 \times 3 - 3 \times 2 - 2 \times 3 + 3 \times 2 \times 2)^{1/2} = \sqrt{9} = 3$
and

$$\langle(3, 2), (-3, 1)\rangle = (3 \times -3 - 2 \times -3 - 3 \times 1 + 3 \times 2 \times 1) = (-9 + 6 - 3 + 6) = 0.$$

(b) Show (by exhibiting an inner product property that fails) that the following formula DOES NOT define an *inner product* on \mathbb{R}^2 :

$\langle(1, 0), (1, 0)\rangle = 1 \times 0 + 0 \times 1 = 0$ but $(1, 0) \neq \mathbf{0}$ so property **4b** fails – positive definiteness. $\langle(x_1, y_1), (x_2, y_2)\rangle = x_1y_2 + x_2y_1$.

11. *Solution.*

(a)

$$\mathbf{u}_1 \cdot \mathbf{u}_1 = \frac{1}{9}((-2)^2 + 2^2 + 1^2 + 0^2) = \frac{9}{9} = 1$$

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \frac{1}{9}((-2) \times 2 + 2 \times 2 + 1 \times 0 + 0 \times 1) = 0$$

$$\mathbf{u}_2 \cdot \mathbf{u}_2 = \frac{1}{9}((2)^2 + 2^2 + 0^2 + 1^2) = \frac{9}{9} = 1$$

Thus the set of vectors is orthogonal and of unit length meaning the set is orthonormal.

(b)

$$\mathbf{v}_1 = \frac{1}{3}(-2, 2, 1, 0) \rightarrow \mathbf{u}_1 = \frac{1}{3}(-2, 2, 1, 0)$$

$$\mathbf{v}_2 = \frac{1}{3}(2, 2, 0, 1) \rightarrow \mathbf{u}_2 = \frac{1}{3}(2, 2, 0, 1)$$

the first two vectors form an orthonormal set from (a)

$$\begin{aligned} \mathbf{v}_3 = (-2, 5, 4, 3) &\rightarrow \mathbf{w}_3 = \mathbf{v}_3 - (\mathbf{u}_1 \cdot \mathbf{v}_3)\mathbf{u}_1 - (\mathbf{u}_2 \cdot \mathbf{v}_3)\mathbf{u}_2 \\ &= (-2, 5, 4, 3) - ((-2, 5, 4, 3) \cdot \frac{1}{3}(-2, 2, 1, 0))\frac{1}{3}(-2, 2, 1, 0) \\ &\quad - ((-2, 5, 4, 3) \cdot \frac{1}{3}(2, 2, 0, 1))\frac{1}{3}(2, 2, 0, 1) \\ &= (-2, 5, 4, 3) - 2(-2, 2, 1, 0) - (2, 2, 0, 1) = (0, -1, 2, 2) \\ &\rightarrow \mathbf{u}_3 = \frac{1}{3}(0, -1, 2, 2) \end{aligned}$$

(c) We use the properties of our orthonormal basis

$$\mathbf{w} = \sum (\mathbf{w} \cdot \mathbf{u}_i)\mathbf{u}_i$$

Now $(0, 3, 3, 3) \cdot \frac{1}{3}(-2, 2, 1, 0) = 3$, $(0, 3, 3, 3) \cdot \frac{1}{3}(2, 2, 0, 1) = 3$, $(0, 3, 3, 3) \cdot \frac{1}{3}(0, -1, 2, 2) = 3$
so

$$(0, 3, 3, 3) = 3\mathbf{u}_1 + 3\mathbf{u}_2 + 3\mathbf{u}_3.$$

12. *Solution.* Let $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be shears along the y -axis and x -axis respectively given by

$$S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

(a) $S \circ T$ has the standard matrix representation

$$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}.$$

(b) The image with respect to $S \circ T$ of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is

$$\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}.$$

(c) As $S \circ T : [1 \ 1]^T \rightarrow [-7 \ 2]^T$ the line $y = x$ maps to $2x + 7y = 0$.

13. Solution.

(a) $\begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix}$.

(b) The transition matrix $P_{\mathcal{B} \rightarrow \mathcal{S}}$ from \mathcal{B} to \mathcal{S} is $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$.

(c) The transition matrix $P_{\mathcal{S} \rightarrow \mathcal{B}}$ from \mathcal{S} to \mathcal{B} is $(P_{\mathcal{B} \rightarrow \mathcal{S}})^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(d) We want $[\mathbf{w}]_{\mathcal{B}}$, $[\mathbf{w}]_{\mathcal{B}} = P_{\mathcal{S} \rightarrow \mathcal{B}}[\mathbf{w}]_{\mathcal{S}} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -7 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 15 \end{bmatrix}$, so

$$\begin{bmatrix} -7 \\ 11 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 15 \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

(e)

$$\begin{aligned} [T]_{\mathcal{B} \rightarrow \mathcal{B}} &= P_{\mathcal{S} \rightarrow \mathcal{B}} [T]_{\mathcal{S} \rightarrow \mathcal{S}} P_{\mathcal{B} \rightarrow \mathcal{S}} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned}$$

(f) $[T(\mathbf{w})]_{\mathcal{B}} = [T]_{\mathcal{B} \rightarrow \mathcal{B}}[\mathbf{w}]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 15 \end{bmatrix} = \begin{bmatrix} 4 \\ -15 \end{bmatrix}$.

14. Solution.

$$\begin{aligned} \det(A - \lambda I) &= \det \left(\begin{bmatrix} 5 - \lambda & 6 \\ -3 & -4 - \lambda \end{bmatrix} \right) \\ &= (5 - \lambda)(-4 - \lambda) - (-3 \times 6) \\ &= \lambda^2 - \lambda - 2 \\ &= 0 \end{aligned}$$

if and only if $\lambda \in \{-1, 2\}$. So the eigenvalues are 2 and -1.

$\lambda = 2$

$$\begin{bmatrix} 5 - \lambda & 6 \\ -3 & -4 - \lambda \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ -3 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

which has non-zero vector $[2 \ -1]^T$ in its solution space. Thus $[2 \ -1]^T$ is an eigenvector for $\lambda = 2$.

$\lambda = -1$

$$\begin{bmatrix} 5 - \lambda & 6 \\ -3 & -4 - \lambda \end{bmatrix} = \begin{bmatrix} 6 & 6 \\ -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

which has non-zero vector $[1 \ -1]^T$ in its solution space. Thus $[1 \ -1]^T$ is an eigenvector for $\lambda = -1$.

15. Solution.

$$(a) \begin{bmatrix} 7 & 10 & -2 \\ 10 & 4 & -8 \\ -2 & -8 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 36 \\ 36 \\ -18 \end{bmatrix} = 18 \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \text{ so } \mathbf{v}_1 \text{ is an eigenvector with eigenvalue 18.}$$

$$\begin{bmatrix} 7 & 10 & -2 \\ 10 & 4 & -8 \\ -2 & -8 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \text{ so } \mathbf{v}_2 \text{ is an eigenvector with eigenvalue 0.}$$

$$\begin{bmatrix} 7 & 10 & -2 \\ 10 & 4 & -8 \\ -2 & -8 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -18 \\ -18 \end{bmatrix} = -9 \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \text{ so } \mathbf{v}_3 \text{ is an eigenvector with eigenvalue -9.}$$

(b)

$$D = P^{-1}AP \text{ where } P = \begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -9 \end{bmatrix}.$$

Note that P^{-1} exists as eigenvectors for different eigenvalues are linearly independent.

(c) We merely need to normalize the orthogonal set of eigenvectors that formed the columns

$$\text{of } P \text{ thus } Q = \frac{1}{3} \begin{bmatrix} 2 & 2 & -1 \\ 2 & -1 & 2 \\ -1 & 2 & 2 \end{bmatrix}.$$

(d) The matrix A is symmetric with distinct eigenvalues thus the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are orthogonal.

(e) $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u}^T A \mathbf{v}$ does not define an inner product on \mathbb{R}^3 as (choosing an eigenvector for a negative eigenvalue)

$$\langle \mathbf{v}_3, \mathbf{v}_3 \rangle = \mathbf{v}_3^T A \mathbf{v}_3 = -9 \mathbf{v}_3^T \mathbf{v}_3 = -81.$$

So the alleged inner product is not positive definite.