Answers to 620-142 exam, semester 2, 2004

1. (a) rank $A = 2$

   (b) Basis for the column space $\left\{ \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ -19 \end{bmatrix} \right\}$.

   (c) dim(row space)=2

   (d) number of rows in A $= 3 >$ dim (row space)=2
   $\Rightarrow$ rows of A are *not* linearly independent.

   (e) The vectors $(1, 2, -5), (3, 8, -19), (1, -2, 3)$ are in the column space of A which has dim=2,
   $\Rightarrow (1, 2, -5), (3, 8, -19), (1, -2, 3)$ span a subspace with dim ≤ 2
   $\Rightarrow$ they do *not* span $\mathbb{R}^3$.

   (f) $(1, -2, 3) = 7(1, 2, -5) - 2(3, 8, -19)$.

   (g) dim (solution space) = # columns - rank = 4-2=2.

   (h) $x_2 = t,\ x_4 = s$
   $x_3 = 2x_4 = 2s,\ x_1 = -2x_2 - 7x_4 = -2t - 7s$

   $$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

   $\Rightarrow$ Solution space has a basis $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$.

2. (a)

   $$A = \{a + bx + cx^2 : c \geq 0\}$$

   is *not* a subspace of $\mathcal{P}_2$ since $x^2 \in A$ but $(-1)x^2 = -x^2 \notin A$ so A is not closed under scalar multiplication.

   (b)

   $$B = \{p(x) : p(1) = 0\}$$

   is a subspace.
   (0) $0 + 0x + 0x^2 \in B$, so B is non-empty.
   (1) If $p(x),\ q(x) \in B$ then $p(1) = q(1) = 0$
   so $(p + q)(1) = p(1) + q(1) = 0 + 0 = 0 \Rightarrow p(x) + q(x) \in B$
   $\Rightarrow\ B$ is closed under addition.
   (2) If $p(x) \in B$ and $\alpha \in \mathbb{R}$, then $(\alpha p)(1) = \alpha p(1) = \alpha \times 0 = 0$.
   $\Rightarrow \alpha p(x) \in B \Rightarrow\ B$ is closed under scalar multiplication.

3. (a)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ (i) the first message must contain an error

(ii) $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} =$ 5th column of $H \Rightarrow$ error in 5th bit

$\Rightarrow$ the first transmitted message is 1100010

the second transmitted message is 0100101 = received message.

(b) dim (solution space) = # columns - rank H= 7 - 3 = 4 $\Rightarrow$ the number of codewords $= 2^4 = 16$.

4. (a) (i) $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y} = (\mathbf{x}^T A \mathbf{y})^T \mathbf{y}^T A^T \mathbf{x} = \mathbf{y}^T A \mathbf{x} = \langle \mathbf{y}, \mathbf{x} \rangle$

$\Rightarrow \langle \cdot, \cdot \rangle$ is symmetric

(ii) $\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 - 2 * x_1 * x_2 + 2 * x_2^2 = (x_1 - x_2)^2 + x_2^2 \geq 0$ and it equals zero exactly when $x_1 = x_2 = 0$

$\Rightarrow \langle \cdot, \cdot \rangle$ is positive definite

Therefore $\langle \cdot, \cdot \rangle$ defines an inner product on $\mathbb{R}^2$.

(b) $u_1^T A u_1 = 1,\ u_2^T A u_2 = 1,\ u_1^T A u_2 = 0$ so it is an orthonormal basis for $\mathbb{R}^2$ with respect to this inner product.

(c) $\|\mathbf{y}\|^2 = y^T A y = 13 \Rightarrow \|\mathbf{y}\| = \sqrt{13}$.

(d) $\mathbf{p} = \langle \mathbf{y}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rangle \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

5. (a) $A^T y = A^T A \bar{u}$

$A^T y = \begin{bmatrix} 2 \\ -12 \end{bmatrix}$ $A^T A = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$

$\Rightarrow \bar{u} = (A^T A)^{-1} A^T y = \dfrac{1}{10} \begin{bmatrix} 18 \\ -26 \end{bmatrix} = \begin{bmatrix} 1.8 \\ -2.6 \end{bmatrix}$

$\Rightarrow$ the line of best fit is $y = 1.8 - 2.6x$

(b) Picture.

(c) When $x = -2$, $y = 1.8 - 2.6(-2) = 7.0$.

6. (a) (i)

$$[S] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [R] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(ii)

$$[S \circ R] = [S][R] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(iii) Images of vertices of T are the columns of

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

So T' is the triangle with vertices $(0,0), (0,1), (1,1)$. (Draw a picture!)

(iv) $S \circ R$ is reflection in the line $y = x$.

(b) (i) $S(\mathbf{b}_1) = \mathbf{0}, S(\mathbf{b}_2) = \mathbf{b}_2 - \mathbf{b}_3, S(\mathbf{b}_3) = -\mathbf{b}_2 + \mathbf{b}_3, S(\mathbf{b}_4) = \mathbf{0}$ so

$$[S]_{\mathcal{B}} = [\, [S(\mathbf{b}_1)]_{\mathcal{B}} \ [S(\mathbf{b}_2)]_{\mathcal{B}} \ [S(\mathbf{b}_3)]_{\mathcal{B}} \ [S(\mathbf{b}_4)]_{\mathcal{B}} \,] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(ii) The dimension of the image of S = rank of $[S]_{\mathcal{B}} = 2$.

7. (a)

$$\det(A-\lambda I) = \begin{vmatrix} \frac{5}{4}-\lambda & -\frac{3}{4} \\ \frac{1}{4} & \frac{1}{4}-\lambda \end{vmatrix} = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda-1)(\lambda-\frac{1}{2}),$$

so the eigenvalues are $\lambda = 1, \frac{1}{2}$.

For $\lambda = 1$, an eigenvector is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. For $\lambda = \frac{1}{2}$, an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) We have $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \; P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

(c) $A^n = PD^nP^{-1}$

(d) $\lim\limits_{n\to\infty} A^n = \lim\limits_{n\to\infty} P \begin{bmatrix} 1 & 0 \\ 0 & (\frac{1}{2})^n \end{bmatrix} P^{-1} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix}$

8. (a)

$$0 = \det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 2 \\ 1 & 2 & -\lambda \end{vmatrix} = -\lambda(\lambda-3)(\lambda+2).$$

$\Rightarrow$ eigenvalues are $\lambda = 0, 3, -2$.

(b) $\lambda = 0$ has unit eigenvector $\frac{1}{\sqrt{6}}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 3$ has unit eigenvector $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = -2$ has unit eigenvector $\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

(c) $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \; Q = \begin{bmatrix} \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} \end{bmatrix}$

9. (a) We have $\lim_{n\to\infty} \dfrac{\log n + e^n}{2^n + n^2} = \lim_{n\to\infty} \dfrac{\frac{\log n}{e^n} + 1}{\frac{2^n}{e^n} + \frac{n^2}{e^n}}$. Since $e = 2.718\ldots > 2$ we have $\lim_{n\to\infty}$ top $= 0+1 = 1$ and $\lim_{n\to\infty}$ bottom $= 0+0 = 0$ by arithmetic of limits and growth rates (or standard limits). Hence a_n diverges (to $+\infty$).

(b) $\dfrac{3n-1}{n+7} \leq \dfrac{3n+\sin(n)}{n+7} \leq \dfrac{3n+1}{n+7}$ for all $n \geq 1$. But $\lim_{n\to\infty} \dfrac{3n-1}{n+7} = \lim_{n\to\infty} \dfrac{3-1/n}{1+7/n} = \dfrac{3-0}{1+0} = 3$ and $\lim_{n\to\infty} \dfrac{3n+1}{n+7} = \lim_{n\to\infty} \dfrac{3+1/n}{1+7/n} = \dfrac{3+0}{1+0} = 3$ by algebra of limits. Hence $\lim_{n\to\infty} \dfrac{3n+\sin(n)}{n+7} = 3$ by the sandwich rule.

(c) This is a limit of type $\dfrac{\infty}{\infty}$ so by l'Hôpital's rule

$$\lim_{n\to\infty} \frac{\log(n)}{\log(n+1)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} = 1.$$

(d) Let $a_{n+1} = \dfrac{2}{3}\left(a_n + \dfrac{1}{a_n^2}\right)$ and $a_1 = 2$.
Now let $\lim_{n\to\infty} a_n = L$. Then taking limits of both sides of the equation $a_{n+1} = \dfrac{2}{3}\left(a_n + \dfrac{1}{a_n^2}\right)$ gives $L = \dfrac{2}{3}\left(L + \dfrac{1}{L^2}\right)$. Hence $L^3 = 2$ and $L = \sqrt[3]{2}$.

10. (a) $\dfrac{1}{2n-1} > \dfrac{1}{2n}$ for all n and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges (harmonic series). Hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by comparison test.

(b) We have an alternating series $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ with $a_n = \dfrac{1}{2n-1}$. This converges by Leibniz's test since (i) $a_n > 0$, (ii) $a_n > a_{n+1}$ for all n, (iii) $\lim_{n\to\infty} a_n = 0$.

(c) We use the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^2}{2^{n+1}} \times \frac{2^n}{n^2} = \frac{1}{2}\left(\frac{n+1}{n}\right)^2 = \frac{1}{2}\left(1 + \frac{1}{n}\right)^2 \to \frac{1}{2} \text{ as } n \to \infty.$$

Since $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \dfrac{1}{2} < 1$, the series converges by the ratio test.

11. (a) Substituting $x = 2t^2$ in the series for $\log(1 + x)$ gives

$$\log(1 + 2t^2) = 2t^2 - \frac{(2t^2)^2}{2} + \frac{(2t^2)^3}{3} - \frac{(2t^2)^4}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n} t^{2n}.$$

This converges for $|2t^2| < 1$, i.e. $|t| < \frac{1}{\sqrt{2}}$.

(b) We can integrate term by term *inside the interval of convergence.* Hence

$$\int_0^x \log(1+2t^2)\, dt = \sum_{n=1}^{\infty} \left(\int_0^x (-1)^{n-1} \frac{2^n}{n} t^{2n}\, dt \right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n(2n+1)} x^{2n+1}.$$

This is valid provided $|x| < \frac{1}{\sqrt{2}}$.

12. (a) For $f(x) = \cosh x$, the degree 2 Taylor polynomial is $p_2(x) = 1 + \dfrac{x^2}{2}$.

(b) $\cosh(\frac{1}{2}) \approx p_2(\frac{1}{2}) = \frac{9}{8} = 1.125$.

(c) From Taylor's theorem

$$\mathit{error} = R_2(\frac{1}{2}) = \frac{f^{(3)}(c)}{3!} \left(\frac{1}{2} \right)^3 = \frac{\sinh(c)}{48}$$

where c is between 0 and $\dfrac{1}{2}$. Since $\sinh(c)$ is an increasing function of c,

$$\sinh(c) \le \sinh(\frac{1}{2}) = \frac{1}{2}(e^{1/2} - e^{-1/2}) < \frac{1}{2} e^{1/2} < \frac{1}{2} \times 2 = 1.$$

Hence $\mathit{error} < \dfrac{1}{48}$. (Or note that $p_2(x) = p_3(x)$ and use $\mathit{error} = R_3(1/2)$ to get a better estimate.)