620-142 Mathematics B
Solutions to Assignment 1

1. (a) If n is even then $n = 2k$ for some integer k. Then $n^3 = 8k^3$ is divisible by 8, since k^3 is an integer.

   (b) Since a is a factor of b, $b = ka$ for some $k \in \mathbb{Z}$. Since b is a factor of c, $c = lb$ for some $l \in \mathbb{Z}$. Hence
$$5b - 3c = 5ka - 3lb = 5ka - 3lka = (5k - 3lk)a$$
   and a is a factor of $5b - 3c$, since $5k - 3lk \in \mathbb{Z}$.

2. (a) (i) The multiplication table for $\mathbb{Z}_9$ is:

×	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8
2	0	2	4	6	8	1	3	5	7
3	0	3	6	0	3	6	0	3	6
4	0	4	8	3	7	2	6	1	5
5	0	5	1	6	2	7	3	8	4
6	0	6	3	0	6	3	0	6	3
7	0	7	5	3	1	8	6	4	2
8	0	8	7	6	5	4	3	2	1

   (ii) $7x+4 \equiv 1 \pmod 9 \Rightarrow 7x \equiv 1-4 \equiv 6 \pmod 9 \Rightarrow x \equiv 6 \pmod 9$ from the multiplication table. (Or: $x = 7^{-1} \times 6 = 4 \times 6 = 6$, using the table to find 7^{-1}.)

   (iii) The elements in $\mathbb{Z}_9$ with no multiplicative inverse are: $0, 3, 6$.
   The multiplicative inverses of the other elements are:
   $1^{-1} = 1$, $2^{-1} = 5$, $4^{-1} = 7$, $5^{-1} = 2$, $7^{-1} = 4$, $8^{-1} = 8$.

   (b) The order of 2 in $\mathbb{Z}_9$ divides $\phi(3^2) = (3-1) \times 3 = 6$ so the order is $1, 2, 3$ or 6. But $2^1 = 2$, $2^2 = 4$, $2^3 = 8 = -1$ so 2 has order 6. (Or: check that $2^1, 2^2, 2^3, 2^4, 2^5 \neq 1$, but $2^6 = 1$.)

3. (a) Using the Euclidean Algorithm:
$$\begin{aligned} 44 &= 3 \times 13 + 5 \\ 13 &= 2 \times 5 + 3 \\ 5 &= 1 \times 3 + 2 \\ 3 &= 1 \times 2 + 1 \\ 2 &= 2 \times 1 + 0 \end{aligned}$$
   Hence $\gcd(44, 13) =$ last non-zero remainder $= 1$.

   (b) Working backwards through the calculation in (a), we obtain
$$\begin{aligned} 1 &= 3 - 1 \times 2 \\ &= 3 - 1 \times (5 - 3) \\ &= 2 \times 3 - 1 \times 5 \\ &= 2 \times (13 - 2 \times 5) - 1 \times 5 \\ &= 2 \times 13 - 5 \times 5 \\ &= 2 \times 13 - 5 \times (44 - 3 \times 13) \\ &= 17 \times 13 - 5 \times 44 \end{aligned}$$
   So $44x + 13y = 1$ where $x = -5$ and $y = 17$.
   (Note: other answers are possible. The general solution in integers is $x = -5 + 13k$, $y = 17 - 44k$ where $k \in \mathbb{Z}$.)

   (c) From (b), $17 \times 13 \equiv 1 \pmod{44}$ hence $13^{-1} = 17$ in $\mathbb{Z}_{44}$.

4. (a) $\phi(77) = \phi(7 \times 11) = 6 \times 10 = 60$
   (b) By Fermat's Little Theorem, $5^{12} = 1$ in $\mathbb{Z}_{13}$. Hence $5^{36} = (5^{12})^3 = 1$.
   (c) We compute 5^{43} in $\mathbb{Z}_{49}$ using the binary powering method from lectures. On the left of the following table we compute powers of 5 in $\mathbb{Z}_{49}$ by repeated squaring, while on the right we are repeatedly dividing 43 by 2, dropping any remainder.

$5^1 = 5$	43
$5^2 = 25 = -24$	21
$5^4 = 576 = -12$	10
$5^8 = 144 = -3$	5
$5^{16} = 9$	2
$5^{32} = 81 = -17$	1

   Hence, in $\mathbb{Z}_{49}$ we have

$$\begin{aligned} 5^{43} &= 5^1 \times 5^2 \times 5^8 \times 5^{32} \\ &= 5 \times -24 \times -3 \times -17 \\ &= -120 \times 51 = -22 \times 2 = -44 = 5. \end{aligned}$$

5. We want to proveerthe statement:

$$S(n): \qquad 1 + 2 \times 2^1 + 3 \times 2^2 + 4 \times 2^3 + \ldots + n \times 2^{n-1} = 1 + (n-1)2^n$$

for all integers $n \geq 1$.

Initial Step: For $n = 1$, the left hand side of $S(1)$ is $1 \times 2^0 = 1$ and the right hand side is $1 + 0 = 1$. So $S(1)$ is true.

Inductive Step: Assume $S(n)$ is true for some $n \geq 1$, i.e.

$$1 + 2 \times 2^1 + 3 \times 2^2 + 4 \times 2^3 + \ldots + n \times 2^{n-1} = 1 + (n-1)2^n.$$

Adding $(n+1) \times 2^n$ to both sides gives:

$$\begin{aligned} & 1 + 2 \times 2^1 + 3 \times 2^2 + 4 \times 2^3 + \ldots + n \times 2^{n-1} + (n+1) \times 2^n \\ = \; & 1 + (n-1)2^n + (n+1) \times 2^n \\ = \; & 1 + 2n \times 2^n \\ = \; & = 1 + n2^{n+1} \end{aligned}$$

so $S(n+1)$ is true.

Hence, by the Principle of Mathematical Induction, $S(n)$ is true for all integers $n \geq 1$.